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2001 J. Phys. A: Math. Gen. 34 3623

(http://iopscience.iop.org/0305-4470/34/17/306)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **34** (2001) 3623–3650 www.iop.org/Journals/ja PII: S0305-4470(01)18466-0

Duality and modular class of a Nambu–Poisson structure

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Received 30 October 2000

Abstract

In this paper we introduce cohomology and homology theories for Nambu–Poisson manifolds. Also we study the relation between the existence of a duality for these theories and the vanishing of a particular Nambu–Poisson cohomology class, the modular class. The case of a regular Nambu–Poisson structure and some singular examples are discussed.

PACS numbers: 0240M, 0220, 0365, 4520J

1. Introduction

Homology and cohomology theories have been shown to be good tools in the study of Poisson geometry, as they have been in other areas of geometry and physics. In particular, a lot of work has been done in the study of Poisson cohomology and Poisson homology (see, for example, [37, 40]). Poisson cohomology (also known as Lichnerowicz–Poisson cohomology) of a Poisson manifold M was introduced by Lichnerowicz [22] as the cohomology of the subcomplex of the Chevalley–Eilenberg complex of the Lie algebra $C^\infty(M, \mathbb{R})$ consisting of the 1-differentiable cochains that are derivations in each argument with respect to the usual product of functions. Poisson cohomology provides a good framework to express deformation and quantization obstructions. On the other hand, Poisson homology (also known as canonical homology) was defined as the homology of the operator boundary δ on differential forms considered geometrically by Koszul [14] and algebraically by Brylinski [4] by taking the classical limit of the Hochschild boundary operator for a quantized Poisson algebra. The notion of Poisson (respectively, symplectic) harmonicity also appears to be very interesting. These cohomology and homology theories can be extended to Lie algebroids, which are algebraic

0305-4470/01/173623+28$30.00

structures of great interest in mathematics and physics [27]. Lie algebroids are a generalization of Lie algebras and tangent bundles and each Poisson manifold has associated a Lie algebroid in a natural way. Recently, a Poincaré-type duality between cohomology and homology theories has been proved by Evens *et al* [12] and Xu [41] using the modular class of the Poisson structure [39]. Furthermore, the rotational of a Poisson structure associated with a volume form (a representative of the modular class) has also been used as a tool for the classification of Poisson structures [10, 18, 23].

The aim of this paper is to introduce similar cohomology and homology theories for Nambu–Poisson structures, as well as the study of a Poincaré-type duality. The concept of a Nambu–Poisson structure was given by Takhtajan [33] in 1994 in order to find an axiomatic formalism for the n-bracket operation

$$\{f_1, \ldots, f_n\} = \det\left(\frac{\partial f_i}{\partial x_j}\right)$$

proposed by Nambu [32] and picking up the idea that in statistical mechanics the basic result is the Liouville theorem, which follows from but does not require Hamiltonian dynamics. A Nambu–Poisson manifold of order n is a manifold M endowed with a skew-symmetric n-bracket of functions $\{\,,\ldots,\,\}$ satisfying the Leibniz rule and the fundamental identity

$$\{f_1, \ldots, f_{n-1}, \{g_1, \ldots, g_n\}\} = \sum_{i=1}^{n}\{g_1, \ldots, \{f_1, \ldots, f_{n-1}, g_i\}, \ldots, g_n\}$$

for all $f_1, \ldots, f_{n-1}, g_1, \ldots, g_n$ C^∞ real-valued functions on M. Note that the n-bracket $\{\,,\ldots,\,\}$ allows us to introduce the Nambu–Poisson n-vector Λ characterized by the relation $\Lambda(\mathrm{d}f_1, \ldots, \mathrm{d}f_n) = \{f_1, \ldots, f_n\}$. The structure is said to be regular if $\Lambda \neq 0$ at every point. Recently, local and global properties of Nambu–Poisson manifolds have been studied [1, 3, 15, 17, 20, 28, 31, 38]. The canonical example of a Nambu–Poisson structure of order n greater than two is that induced by a volume form on an oriented manifold of dimension n. In fact, a Nambu–Poisson manifold of order n, $n \geqslant 3$, admits a generalized foliation (the characteristic foliation) whose leaves are either points or n-dimensional manifolds endowed with a volume Nambu–Poisson structure. A strong effort is being made in order to understand the geometry of Nambu–Poisson structures, and also to understand the Nambu mechanics (see, for example, [5, 7, 8]).

Recently, the authors have defined in [21] the notion of a Leibniz algebroid in the same way as for the case of a Lie algebroid but bearing in mind the concept of Leibniz algebra [24, 25]. A Leibniz algebra is a real vector space $\mathfrak{g}$ endowed with a $\mathbb{R}$-bilinear mapping $\{\,,\,\}$ satisfying the Leibniz identity

$$\{a_1, \{a_2, a_3\}\} - \{\{a_1, a_2\}, a_3\} - \{a_2, \{a_1, a_3\}\} = 0$$

for $a_1, a_2, a_3 \in \mathfrak{g}$. If the bracket is skew-symmetric we recover the notion of a Lie algebra. In [21], it was shown that each Nambu–Poisson manifold (M, Λ) of order n, with $n \geqslant 3$, has associated a Leibniz algebroid, consisting in the vector bundle $\Lambda^{n-1}(T^*M) \longrightarrow M$ whose space of sections $\Omega^{n-1}(M)$ has a Leibniz algebra structure with bracket

$$[\![\alpha, \beta]\!] = \mathcal{L}_{\#_{n-1}(\alpha)}\beta + (-1)^n(i(\mathrm{d}\alpha)\Lambda)\beta$$

and a vector bundle homomorphism $\#_{n-1} : \Lambda^{n-1}(T^*M) \longrightarrow TM$ given by $\#_{n-1}(\beta) = i(\beta)\Lambda$, which provides a Leibniz algebra homomorphism between the spaces of sections. The Leibniz algebroid $(\Lambda^{n-1}(T^*M), [\![\,,\,]\!], \#_{n-1})$ allows us to introduce the Leibniz algebroid cohomology. However, this cohomology has infinite degrees and thus a Poincaré-type duality, with some homology theory, is not possible.

In this paper, in order to obtain a cohomology theory for Nambu–Poisson manifolds without the above-mentioned problems, we begin by showing in section 3 a Lie algebra structure associated with a Nambu–Poisson manifold (M, Λ). In fact, we prove that the centre of the Leibniz algebra $(\Omega^{n-1}(M), [\![\, ,\,]\!])$ is the $C^\infty(M, \mathbb{R})$-module $\ker \#_{n-1} = \{\alpha \in \Omega^{n-1}(M)/\#_{n-1}(\alpha) = 0\}$ and thus the quotient space $\frac{\Omega^{n-1}(M)}{\ker \#_{n-1}}$ is a Lie algebra. Moreover, if the Nambu–Poisson structure is regular, $\frac{\Omega^{n-1}(M)}{\ker \#_{n-1}}$ is the space of sections of the vector bundle $\frac{\Lambda^{n-1}(T^*M)}{\ker \#_{n-1}} \to M$ and this is a Lie algebroid. As a consequence of the above results, we introduce in section 4 a cohomology theory for a Nambu–Poisson manifold (M, Λ). The resultant cohomology, called Nambu–Poisson cohomology, is defined as the cohomology of the Lie algebra $\frac{\Omega^{n-1}(M)}{\ker \#_{n-1}}$ relative to a certain representation. If the structure is regular, the Nambu–Poisson cohomology is just the Lie algebroid cohomology of $\frac{\Lambda^{n-1}(T^*M)}{\ker \#_{n-1}} \to M$. So, we can think that for a Nambu–Poisson structure there exists associated a kind of 'singular' Lie algebroid structure and the corresponding cohomology. Also in section 4, we observe that the characteristic foliation of a Nambu–Poisson manifold allows us to introduce the foliated cohomology which, in the regular case, coincides with the usual foliated cohomology defined for regular foliations [13, 19, 35, 36]. Furthermore, in this last case, we prove that the foliated cohomology is isomorphic to the Nambu–Poisson cohomology. After this paper was finished, some computations of the Nambu–Poisson cohomology have been done in [29].

Section 5 is devoted to the introduction of the canonical Nambu–Poisson homology on an oriented Nambu–Poisson manifold. If M is an oriented manifold one can consider, in a natural way, a homology complex whose k-chains are the k-vectors on M, the homology operator on vector fields is the divergence with respect to a volume and the resultant homology is dual to the de Rham cohomology. The canonical Nambu–Poisson homology complex of an oriented Nambu–Poisson manifold (M, Λ) is a subcomplex of this homology complex. In fact, if (M, Λ) is regular, the k-chains in the canonical Nambu–Poisson homology complex are the k-vectors on M which are tangent to the characteristic foliation.

In section 6, we study the relation between the vanishing of the modular class of an oriented Nambu–Poisson manifold (M, Λ) and the existence of a duality between the homology and cohomology theories introduced in the above sections. The modular tensor of M was introduced independently in [9, 21]. Recently, Dufour and Zhitomirskii [11] have used this tensor in order to give a classification of quadratic integrable 1-forms. The modular tensor defines a cohomology class (the modular class) in the Leibniz algebroid cohomology which is null in some neighbourhood of any regular point (see [21]). An example of a singular Nambu–Poisson structure with non-null modular class was also exhibited in [21]. Now, if M is an oriented regular Nambu–Poisson manifold of order n ($n \geqslant 3$) then, in section 6, we prove that the modular class of M is null if and only if there exists a basic volume with respect to the characteristic foliation. Using this result, we obtain some interesting examples of regular Nambu–Poisson structures with non-null modular class. Next, we show that the vanishing of the modular class implies the existence of a duality between the foliated cohomology of M and the homology of a subcomplex of the canonical Nambu–Poisson homology complex of M. Thus, if (M, Λ) is regular and there exists a basic volume with respect to the characteristic foliation of M, we conclude that there is a duality between the Nambu–Poisson cohomology and the canonical Nambu–Poisson homology of M.

Finally, in section 7, we study a particular example, namely, a singular Nambu–Poisson structure of order three on $\mathbb{R}^3$. We prove that there is no duality between the canonical Nambu–Poisson homology and the Nambu–Poisson cohomology and that this last cohomology is not isomorphic to the foliated cohomology.

2. Preliminaries

All the manifolds considered in this paper are assumed to be connected.

2.1. Nambu–Poisson structures

Let M be a differentiable manifold of dimension m. Denote by $\mathfrak{X}(M)$ the Lie algebra of vector fields on M, by $C^\infty(M, \mathbb{R})$ the algebra of C^∞ real-valued functions on M, by $\Omega^k(M)$ the space of k-forms on M and by $\mathcal{V}^k(M)$ the space of k-vectors.

A Nambu–Poisson bracket of order n ($n \leqslant m$) on M (see [33]) is an n-linear mapping $\{ , \ldots , \} : C^\infty(M, \mathbb{R}) \times \cdots^{(n} \cdots \times C^\infty(M, \mathbb{R}) \to C^\infty(M, \mathbb{R})$ satisfying the following properties:

(1) *Skew-symmetry:*

$$\{f_1, \ldots , f_n\} = (-1)^{\varepsilon(\sigma)}\{f_{\sigma(1)}, \ldots , f_{\sigma(n)}\}$$

for all $f_1, \ldots , f_n \in C^\infty(M, \mathbb{R})$ and $\sigma \in Symm(n)$, where $Symm(n)$ is a symmetric group of n elements and $\varepsilon(\sigma)$ is the parity of the permutation σ.

(2) *Leibniz rule:*

$$\{f_1g_1, f_2, \ldots , f_n\} = f_1\{g_1, f_2, \ldots , f_n\} + g_1\{f_1, f_2, \ldots , f_n\}$$

for all $f_1, \ldots , f_n, g_1 \in C^\infty(M, \mathbb{R})$.

(3) *Fundamental identity*:

$$\{f_1, \ldots , f_{n-1}, \{g_1, \ldots , g_n\}\} = \sum_{i=1}^{n}\{g_1, \ldots , \{f_1, \ldots , f_{n-1}, g_i\}, \ldots , g_n\}$$

for all $f_1, \ldots , f_{n-1}, g_1, \ldots , g_n$ functions on M.

Given a Nambu–Poisson bracket, we can define a skew-symmetric tensor Λ of type $(n, 0)$ (n-vector) as follows:

$$\Lambda(\mathrm{d}f_1, \ldots , \mathrm{d}f_n) = \{f_1, \ldots , f_n\}$$

for $f_1, \ldots , f_n \in C^\infty(M, \mathbb{R})$. The pair (M, Λ) is called *a Nambu–Poisson manifold of order n*.

Let (M, Λ) be a Nambu–Poisson manifold of order n and k be an integer with $k \leqslant n$.

If $\Lambda^k(T^*M)$ (respectively, $\Lambda^{n-k}(TM)$) denotes the vector bundle of the k-forms (respectively, $(n - k)$-vectors) then Λ induces a homomorphism of vector bundles $\#_k : \Lambda^k(T^*M) \to \Lambda^{n-k}(TM)$ by defining

$$\#_k(\beta) = i(\beta)\Lambda(x) \tag{2.1}$$

for $\beta \in \Lambda^k(T^*_xM)$ and $x \in M$, where $i(\beta)$ is the contraction by β. Denote also by $\#_k$ the homomorphism of $C^\infty(M, \mathbb{R})$-modules from the space $\Omega^k(M)$ onto the space $\mathcal{V}^{n-k}(M)$ given by

$$\#_k(\alpha)(x) = \#_k(\alpha(x)) \tag{2.2}$$

for all $\alpha \in \Omega^k(M)$ and $x \in M$.

Remark 2.1. It is clear that the mapping $\#_k : \Omega^k(M) \to \mathcal{V}^{n-k}(M)$ induces an isomorphism of $C^\infty(M, \mathbb{R})$-modules $\overline{\#_k} : \frac{\Omega^k(M)}{\ker \#_k} \to \#_k(\Omega^k(M))$ defined by

$$\overline{\#_k}([\alpha]) = \#_k(\alpha) \tag{2.3}$$

for $[\alpha] \in \frac{\Omega^k(M)}{\ker \#_k}$.

If $f_1, \ldots, f_{n-1}$ are $n-1$ functions on M, we define a vector field

$$X_{f_1 \cdots f_{n-1}} = \#_{n-1}(\mathrm{d} f_1 \wedge \cdots \wedge \mathrm{d} f_{n-1}) \tag{2.4}$$

which is called the *Hamiltonian vector field* associated with the Hamiltonian functions $f_1, \ldots, f_{n-1}$.

From the fundamental identity, it follows that the Hamiltonian vector fields are infinitesimal automorphisms of Λ, i.e.

$$\mathcal{L}_{X_{f_1 \cdots f_{n-1}}} \Lambda = 0 \tag{2.5}$$

for all $f_1, \ldots, f_{n-1} \in C^\infty(M, \mathbb{R})$.

Example 2.2. Let M be an oriented m-dimensional manifold and choose a volume form ν_M on M. Then, we can consider the following Nambu–Poisson bracket $\{, \ldots, \}$ defined by the formula

$$\mathrm{d} f_1 \wedge \cdots \wedge \mathrm{d} f_m = \{f_1, \ldots, f_m\} \nu_M.$$

In this case the homomorphisms $\#_k$ are isomorphisms, for all $k \leqslant m$ (see [15]).

The following theorem describes the local structure of the Nambu–Poisson brackets of order n, with $n \geqslant 3$.

Theorem 2.3 (See [1,15,20,28,31]). *Let M be a differentiable manifold of dimension m. The n-vector Λ, $n \geqslant 3$, defines a Nambu–Poisson structure on M if and only if for all $x \in M$ with $\Lambda(x) \neq 0$, there exist local coordinates $(x^1, \ldots, x^n, x^{n+1}, \ldots, x^m)$ around x such that*

$$\Lambda = \frac{\partial}{\partial x^1} \wedge \cdots \wedge \frac{\partial}{\partial x^n}.$$

A point x of a Nambu–Poisson manifold (M, Λ) of order $n \geqslant 3$ is said to be *regular* if $\Lambda(x) \neq 0$. If every point of M is regular then the Nambu–Poisson manifold (M, Λ) is said to be *regular*.

Let (M, Λ) be a Nambu–Poisson manifold of order n, with $n \geqslant 3$, and consider *the characteristic distribution* $\mathcal{D}$ on M, given by

$$\begin{aligned} x \in M \to \mathcal{D}(x) &= \#_{n-1}(\Lambda^{n-1}(T_x^* M)) \\ &= \left\langle \{X_{f_1 \cdots f_{n-1}}(x) / f_1, \ldots, f_{n-1} \in C^\infty(M, \mathbb{R})\} \right\rangle \subseteq T_x M. \end{aligned} \tag{2.6}$$

Then, $\mathcal{D}$ defines a generalized foliation on M whose leaves are either points or n-dimensional manifolds endowed with a Nambu–Poisson structure coming from a volume form (see [20]).

Remark 2.4. Let (M, Λ) be an m-dimensional regular Nambu–Poisson manifold of order n, with $n \geqslant 3$. From theorem 2.3, we deduce.

(i) $\mathcal{D}$ defines a foliation on M of dimension n.

(ii) For all $k \in \{0, \ldots, n\}$, $\ker \#_k$ (respectively, $\#_k(\Lambda^k(T^*M))$) is a vector subbundle of $\Lambda^k(T^*M) \to M$ (respectively, $\Lambda^{n-k}(TM) \to M$) of rank $\binom{m}{k} - \binom{n}{k}$ (respectively, $\binom{n}{k}$) and the homomorphism $\#_k : \Lambda^k(T^*M) \to \Lambda^{n-k}(TM)$ induces an isomorphism of vector bundles

$$\overline{\#_k} : \frac{\Lambda^k(T^*M)}{\ker \#_k} \to \#_k(\Lambda^k(T^*M)).$$

The notation $\overline{\#_k}$ is justified by the following fact. The space of the C^∞-differentiable sections of $\frac{\Lambda^k(T^*M)}{\ker \#_k} \to M$ (respectively, $\#_k(\Lambda^k(T^*M)) \to M$) can be identified with $\frac{\Omega^k(M)}{\ker \#_k}$ (respectively, $\#_k(\Omega^k(M))$) in such a sense that the corresponding isomorphism of $C^\infty(M, \mathbb{R})$-modules induced by $\overline{\#_k}$ is just the mapping $\overline{\#_k} : \frac{\Omega^k(M)}{\ker \#_k} \to \#_k(\Omega^k(M))$ given by (2.3).

(iii) The C^∞-differentiable sections of the vector bundle $\#_k(\Lambda^k(T^*M)) \to M$ are the $(n-k)$-vectors on M which are tangent to $\mathcal{D}$. We recall that an $(n-k)$-vector P on M is tangent to $\mathcal{D}$ if

$$i(\alpha(x))(P(x)) = 0$$

for all $x \in M$ and for all $\alpha(x) \in \mathcal{D}^0(x)$, where $\mathcal{D}^0(x)$ is the annihilator of $\mathcal{D}(x)$ in T_x^*M. Note that $\mathcal{D}^0(x) = \ker(\#_{1|T_x^*M})$, for all $x \in M$.

2.2. *The Leibniz algebroid associated with a Nambu–Poisson structure*

In [21] we have introduced the notion of a Leibniz algebroid, a natural generalization of the notion of a Lie algebroid, and we have proved that every Nambu–Poisson manifold has associated a canonical Leibniz algebroid. Next, we will describe this structure.

First, we recall the definition of real Leibniz algebra (see [6, 24–26]). A *Leibniz algebra structure* on a real vector space $\mathfrak{g}$ is a $\mathbb{R}$-bilinear map $\{\, , \}: \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying the *Leibniz identity*, that is,

$$\{a_1, \{a_2, a_3\}\} - \{\{a_1, a_2\}, a_3\} - \{a_2, \{a_1, a_3\}\} = 0$$

for $a_1, a_2, a_3 \in \mathfrak{g}$. In such a case, the pair $(\mathfrak{g}, \{\, , \})$ is called a *Leibniz algebra*.

Moreover, if the skew-symmetric condition is required then $(\mathfrak{g}, \{\, , \})$ is a Lie algebra. In this sense, a Leibniz algebra is a non-commutative version of a Lie algebra.

The notion of Leibniz algebroid can be introduced in the same way as that of Lie algebroid.

Definition 2.5. *A Leibniz algebroid structure on a differentiable vector bundle $\pi : E \to M$ is a pair that consists of a Leibniz algebra structure $[\![\, ,]\!]$ on the space $\Gamma(E)$ of the global cross sections of $\pi : E \longrightarrow M$ and a vector bundle morphism $\varrho : E \to TM$, called the anchor map, such that the induced map $\varrho : \Gamma(E) \longrightarrow \Gamma(TM) = \mathfrak{X}(M)$ satisfies the following relations:*

(a) $\varrho[\![s_1, s_2]\!] = [\varrho(s_1), \varrho(s_2)]$,
(b) $[\![s_1, fs_2]\!] = f[\![s_1, s_2]\!] + \varrho(s_1)(f)s_2$

for all $s_1, s_2 \in \Gamma(E)$ and $f \in C^\infty(M, \mathbb{R})$.

A triple $(E, [\![\, ,]\!], \varrho)$ is called a Leibniz algebroid over M.

Every Lie algebroid over a manifold M is trivially a Leibniz algebroid. In fact, a Leibniz algebroid $(E, [\![\, ,]\!], \varrho)$ over M is a Lie algebroid if and only if the Leibniz bracket $[\![\, ,]\!]$ on $\Gamma(E)$ is skew-symmetric.

Now, let (M, Λ) be a Nambu–Poisson manifold of order n, $n \geqslant 3$, and $\mathcal{L}$ the Lie derivative operator on M. The Leibniz algebroid attached to M is just the triple $(\Lambda^{n-1}(T^*M), [\![\,,\,]\!], \#_{n-1})$, where $[\![\,,\,]\!] : \Omega^{n-1}(M) \times \Omega^{n-1}(M) \to \Omega^{n-1}(M)$ is the bracket of $(n-1)$-forms defined by

$$[\![\alpha, \beta]\!] = \mathcal{L}_{\#_{n-1}(\alpha)}\beta + (-1)^n \#_n(\mathrm{d}\alpha)\beta \tag{2.7}$$

for all $\alpha, \beta \in \Omega^{n-1}(M)$. In particular, we have that

$$\#_{n-1}([\![\alpha, \beta]\!]) = [\#_{n-1}(\alpha), \#_{n-1}(\beta)] \tag{2.8}$$

for all $\alpha, \beta \in \Omega^{n-1}(M)$.

Moreover, in [21] it was proved that the only non-null Nambu–Poisson structures of order greater than two on an oriented manifold M of dimension m such that its Leibniz algebroid is a Lie algebroid are those defined by non-null m-vectors.

Let $(E, [\![\,,\,]\!], \varrho)$ be a Leibniz algebroid over a manifold M. For every $k \in \mathbb{N}$, we consider the vector space

$$C^k(\Gamma(E); C^\infty(M, \mathbb{R})) = \{c^k : \Gamma(E) \times \cdots^{(k} \cdots \times \Gamma(E) \to C^\infty(M, \mathbb{R}) / c^k \text{ is } k\text{-linear}\}$$

and the operator $\partial : C^k(\Gamma(E); C^\infty(M, \mathbb{R})) \to C^{k+1}(\Gamma(E); C^\infty(M, \mathbb{R}))$ defined by

$$\begin{aligned} \partial c^k(s_0, \ldots, s_k) &= \sum_{i=0}^{k} (-1)^i \varrho(s_i)(c^k(s_0, \ldots, \widehat{s_i}, \ldots, s_k)) \\ &\quad + \sum_{0 \leqslant i < j \leqslant k} (-1)^{i-1} c^k(s_0, \ldots, \widehat{s_i}, \ldots, s_{j-1}, [\![s_i, s_j]\!], s_{j+1}, \ldots, s_k) \end{aligned}$$

for $c^k \in C^k(\Gamma(E); C^\infty(M, \mathbb{R}))$ and $s_0, \ldots, s_k \in \Gamma(E)$.

Then, it follows that $\partial^2 = 0$. The resultant cohomology is called the *Leibniz algebroid cohomology of E*. This cohomology also can be described as that defined by the representation

$$\Gamma(E) \times C^\infty(M, \mathbb{R}) \to C^\infty(M, \mathbb{R}) \qquad (s, f) \mapsto \varrho(s)(f).$$

The definition of the cohomology of a Leibniz algebra relative to a representation can be found in [24–26].

Note that if $c^k \in C^k(\Gamma(E); C^\infty(M, \mathbb{R}))$ is skew-symmetric (respectively, $C^\infty(M, \mathbb{R})$-linear) then, in general, ∂c^k is not skew-symmetric (respectively, $C^\infty(M, \mathbb{R})$-linear) (for more details, see [21]).

Nevertheless, if $(E, [\![\,,\,]\!], \varrho)$ is a Lie algebroid and $c^k \in C^k(\Gamma(E); C^\infty(M, \mathbb{R}))$ is skew-symmetric and $C^\infty(M, \mathbb{R})$-linear then ∂c^k is also skew-symmetric and $C^\infty(M, \mathbb{R})$-linear. Thus, in this case, we can consider the subcomplex of $(C^*(\Gamma(E); C^\infty(M, \mathbb{R})), \partial^*)$ that consists of the skew-symmetric $C^\infty(M, \mathbb{R})$-linear cochains. The cohomology of this subcomplex is just *the Lie algebroid cohomology of E* (see [27]).

Remark 2.6. Let (M, Λ) be a Nambu–Poisson manifold of order n, with $n \geqslant 3$, and $(\Lambda^{n-1}(T^*M), [\![\,,\,]\!], \#_{n-1})$ the corresponding Leibniz algebroid. Now, the Leibniz algebroid cohomology operator is given by

$$\begin{aligned} \partial c^k(\alpha_0, \ldots, \alpha_k) &= \sum_{i=0}^{k} (-1)^i \#_{n-1}(\alpha_i)(c^k(\alpha_0, \ldots, \widehat{\alpha_i}, \ldots, \alpha_k)) \\ &\quad + \sum_{0 \leqslant i < j \leqslant k} (-1)^{i-1} c^k(\alpha_0, \ldots, \widehat{\alpha_i}, \ldots, \alpha_{j-1}, [\![\alpha_i, \alpha_j]\!], \alpha_{j+1}, \ldots, \alpha_k) \end{aligned} \tag{2.9}$$

for all $c^k \in C^k(\Omega^{n-1}(M); C^\infty(M, \mathbb{R}))$ and $\alpha_0, \ldots, \alpha_k \in \Omega^{n-1}(M)$.

3. A Lie algebra associated with a Nambu–Poisson manifold

If $(\mathfrak{g}, [\, ,\,])$ is a Leibniz algebra, we define its *centre*, $Z(\mathfrak{g})$, as the kernel of the adjoint representation

$$ad : \mathfrak{g} \to \operatorname{End}(\mathfrak{g}) \qquad x \mapsto [x, \cdot\,].$$

It is easy to prove that $\mathfrak{g}/Z(\mathfrak{g})$ endowed with the induced bracket is a Lie algebra (see [6]).

In the particular case of a Nambu–Poisson manifold (M, Λ) of order $n \geqslant 3$, we have that the centre of the Leibniz algebra $(\Omega^{n-1}(M), [\![\, ,\,]\!])$ is the space

$$Z(\Omega^{n-1}(M)) = \{\alpha \in \Omega^{n-1}(M) \;/\; [\![\alpha, \beta]\!] = 0, \;\forall \beta \in \Omega^{n-1}(M)\}$$

and that $(\Omega^{n-1}(M)/Z(\Omega^{n-1}(M)), \widetilde{[\![\, ,\,]\!]})$ is a Lie algebra, where

$$\widetilde{[\![\, ,\,]\!]} : \Omega^{n-1}(M)/Z(\Omega^{n-1}(M)) \times \Omega^{n-1}(M)/Z(\Omega^{n-1}(M)) \to \Omega^{n-1}(M)/Z(\Omega^{n-1}(M))$$

is the bracket given by

$$\widetilde{[\![[\alpha], [\beta]]\!]} = [[\![\alpha, \beta]\!]] \tag{3.1}$$

for all $[\alpha], [\beta] \in \Omega^{n-1}(M)/Z(\Omega^{n-1}(M))$.

The next result gives an explicit description of the centre of $(\Omega^{n-1}(M), [\![\, ,\,]\!])$.

Proposition 3.1. *Let (M, Λ) be an m-dimensional Nambu–Poisson manifold of order n, with $n \geqslant 3$. Then, the centre of the algebra $(\Omega^{n-1}(M), [\![\, ,\,]\!])$ is the $C^\infty(M, \mathbb{R})$-module*

$$\ker \#_{n-1} = \{\alpha \in \Omega^{n-1}(M) / \#_{n-1}(\alpha) = 0\}.$$

Proof. If α is an $(n-1)$-form on M such that $\#_{n-1}(\alpha) = 0$ then, from (2.7), it follows that

$$[\![\alpha, \beta]\!] = (-1)^n \#_n(\mathrm{d}\alpha)\beta \tag{3.2}$$

for all $\beta \in \Omega^{n-1}(M)$.

On the other hand, using a result proved in [21] (see relation (3.3) in [21]), we have that

$$0 = \mathcal{L}_{\#_{n-1}(\alpha)}\Lambda = (-1)^n \#_n(\mathrm{d}\alpha)\Lambda.$$

Thus, we deduce that $\#_n(\mathrm{d}\alpha) = 0$. Consequently, $[\![\alpha, \beta]\!] = 0$ (see (3.2)).

Conversely, suppose that α is an $(n-1)$-form on M such that

$$[\![\alpha, \beta]\!] = 0 \qquad \text{for all} \quad \beta \in \Omega^{n-1}(M). \tag{3.3}$$

In order to prove that $\#_{n-1}(\alpha)(x) = 0$, for all $x \in M$, we distinguish two cases.

(i) If $\Lambda(x) = 0$, it is obvious that $\#_{n-1}(\alpha)(x) = 0$.

(ii) If $\Lambda(x) \neq 0$ then, using theorem 2.3, we have that there exist local coordinates $(x^1, \ldots, x^n, x^{n+1}, \ldots, x^m)$ in a connected open neighbourhood U of x such that

$$\Lambda = \frac{\partial}{\partial x^1} \wedge \cdots \wedge \frac{\partial}{\partial x^n}. \tag{3.4}$$

Now, the $(n-1)$-form α on U can be written as follows:

$$\alpha = \sum_{i=1}^{n} (-1)^{n-i} \alpha_i \, \mathrm{d}x^1 \wedge \cdots \wedge \widehat{\mathrm{d}x^i} \wedge \cdots \wedge \mathrm{d}x^n + \alpha' \tag{3.5}$$

where $\alpha_i \in C^\infty(U, \mathbb{R})$ and α' is an $(n-1)$-form on U satisfying the condition $\#_{n-1}(\alpha') = 0$.

Note that on U

$$\#_{n-1}(\alpha) = \sum_{i=1}^{n} \alpha_i \frac{\partial}{\partial x^i}. \tag{3.6}$$

On the other hand, from (2.8), (3.3), (3.4) and (3.6), we obtain that, for all $j \in \{1, \ldots, n\}$,

$$0 = \#_{n-1}([\![\alpha, (-1)^{n-j} \, \mathrm{d}x^1 \wedge \cdots \wedge \widehat{\mathrm{d}x^j} \wedge \cdots \wedge \mathrm{d}x^n]\!]) = \left[\#_{n-1}(\alpha), \frac{\partial}{\partial x^j}\right] = -\sum_{i=1}^{n} \frac{\partial \alpha_i}{\partial x^j} \frac{\partial}{\partial x^i}.$$

Consequently,

$$\frac{\partial \alpha_i}{\partial x^j} = 0 \qquad \text{for all} \quad i, j \in \{1, \ldots, n\}. \tag{3.7}$$

This implies that (see (3.4) and (3.5)) on U, we have

$$\#_n(\mathrm{d}\alpha) = 0. \tag{3.8}$$

Moreover, we shall see that $\mathrm{d}\alpha_i = 0$, for all $i \in \{1, \ldots, n\}$. Indeed, consider the $(n-1)$-forms $\beta = \mathrm{d}x^1 \wedge \cdots \wedge \widehat{\mathrm{d}x^j} \wedge \cdots \wedge \mathrm{d}x^n$, for all j. Since $[\![\alpha, \beta]\!] = 0$, using (3.6) and (3.8), we obtain

$$0 = [\![\alpha, \beta]\!] = \mathcal{L}_{\#_{n-1}(\alpha)} \beta = \sum_{i=1}^{n} \mathrm{d}\alpha_i \wedge i\left(\frac{\partial}{\partial x^i}\right)(\mathrm{d}x^1 \wedge \cdots \wedge \widehat{\mathrm{d}x^j} \wedge \cdots \wedge \mathrm{d}x^n).$$

Thus, $\frac{\partial \alpha_i}{\partial x^k} = 0$ for all $k \in \{n+1, \ldots, m\}$ and for all $i \in \{1, \ldots, n\}$. This fact and (3.7) imply that $\mathrm{d}\alpha_i = 0$, that is, α_i is a real constant, for all $i \in \{1, \ldots, n\}$.

Next, we will prove that $\alpha_i = 0$ for all $i \in \{1, \ldots, n\}$. We consider the $(n-1)$-form β' on U given by

$$\beta' = x^j \, \mathrm{d}x^1 \wedge \cdots \wedge \widehat{\mathrm{d}x^j} \wedge \cdots \wedge \mathrm{d}x^n.$$

Using (2.7), (3.6), (3.8) and the fact that α_i is constant, we have that

$$0 = [\![\alpha, \beta']\!] = \alpha_j \, \mathrm{d}x^1 \wedge \cdots \wedge \widehat{\mathrm{d}x^j} \wedge \cdots \wedge \mathrm{d}x^n.$$

Therefore,

$$\alpha_j = 0 \qquad \text{for all} \quad j \in \{1, \ldots, n\}.$$

Finally, from (3.6) we conclude that $\#_{n-1}(\alpha) = 0$ on U. In particular,

$$\#_{n-1}(\alpha)(x) = 0. \qquad \square$$

Hence, if (M, Λ) is an m-dimensional Nambu–Poisson manifold of order n, the quotient space

$$\Omega^{n-1}(M)/Z(\Omega^{n-1}(M)) = \Omega^{n-1}(M)/\ker \#_{n-1}$$

is a $C^\infty(M,\mathbb{R})$-module endowed with a skew-symmetric bracket $[\![\,,\,]\!]^{\sim}$ given by (3.1) which satisfies the Jacobi identity and the following property:

$$[\![[\alpha], f[\beta]]\!]^{\sim} = f[\![[\alpha],[\beta]]\!]^{\sim} + \#_{n-1}(\alpha)(f)[\beta] \tag{3.9}$$

for all $[\alpha], [\beta] \in \Omega^{n-1}(M)/\ker\#_{n-1}$ and $f \in C^\infty(M,\mathbb{R})$.

Furthermore, using (2.8) we obtain that the mapping $\widetilde{\#_{n-1}} : \Omega^{n-1}(M)/\ker\#_{n-1} \to \mathfrak{X}(M)$ defined by

$$\widetilde{\#_{n-1}}([\alpha]) = \#_{n-1}(\alpha) \tag{3.10}$$

induces a homomorphism of Lie algebras between $(\Omega^{n-1}(M)/\ker\#_{n-1}, [\![\,,\,]\!]^{\sim})$ and $(\mathfrak{X}(M), [\,,\,])$.

Remark 3.2. Let (M,Λ) be a regular Nambu–Poisson manifold of order n, with $n \geqslant 3$.

(i) Using the above facts and remark 2.4, we deduce that the triple

$$\left(\frac{\Lambda^{n-1}(T^*M)}{\ker\#_{n-1}}, [\![\,,\,]\!]^{\sim}, \widetilde{\#_{n-1}}\right)$$

is a Lie algebroid over M.

(ii) If $\mathcal{F}$ is a foliation on a manifold N and $F = \bigcup_{x\in N}\mathcal{F}(x) \to N$ is the corresponding vector subbundle of TN then the triple $(F, [\,,\,], i)$ is a Lie algebroid over N, where $[\,,\,]$ is the usual Lie bracket of vector fields and $i : F \to TN$ is the inclusion.

(iii) If $\mathcal{D}$ is the characteristic foliation of M, then the Lie algebroids $(\bigcup_{x\in M}\mathcal{D}(x) = \#_{n-1}(\Lambda^{n-1}(T^*M)), [\,,\,], i)$, $(\frac{\Lambda^{n-1}(T^*M)}{\ker\#_{n-1}}, [\![\,,\,]\!]^{\sim}, \widetilde{\#_{n-1}})$ are isomorphic (see remark 2.4).

4. The Nambu–Poisson cohomology and the foliated cohomology

Let (M,Λ) be a Nambu–Poisson manifold of order n, $n \geqslant 3$. According to the precedent section, the quotient space $\frac{\Omega^{n-1}(M)}{\ker\#_{n-1}}$ endowed with the bracket $[\![\,,\,]\!]^{\sim}$ given by (3.1) is a Lie algebra.

Moreover, using (2.8), we deduce that $C^\infty(M,\mathbb{R})$ is a $(\Omega^{n-1}(M)/\ker\#_{n-1})$-module relative to the representation:

$$\Omega^{n-1}(M)/\ker\#_{n-1} \times C^\infty(M,\mathbb{R}) \to C^\infty(M,\mathbb{R}) \qquad ([\alpha], f) \mapsto [\alpha]f = (\#_{n-1}(\alpha))(f).$$

Thus, one can consider the skew-symmetric cohomology complex

$$\left(C^*(\Omega^{n-1}(M)/\ker\#_{n-1}; C^\infty(M,\mathbb{R})) = \bigoplus_k C^k(\Omega^{n-1}(M)/\ker\#_{n-1}; C^\infty(M,\mathbb{R})), \tilde{\partial}\right)$$

where the space of the k-cochains $C^k(\Omega^{n-1}(M)/\ker\#_{n-1}; C^\infty(M,\mathbb{R}))$ consists of skew-symmetric $C^\infty(M,\mathbb{R})$-linear mappings

$$c^k : (\Omega^{n-1}(M)/\ker\#_{n-1}) \times \cdots^{(k}\cdots \times (\Omega^{n-1}(M)/\ker\#_{n-1}) \to C^\infty(M,\mathbb{R})$$

and the cohomology operator $\tilde{\partial}$ is given by

$$\begin{aligned}\tilde{\partial}c^k([\alpha_0],\ldots,[\alpha_k]) &= \sum_{i=0}^{k}(-1)^i(\#_{n-1}(\alpha_i))(c^k([\alpha_0],\ldots,\widehat{[\alpha_i]},\ldots,[\alpha_k]))\\ &\quad+\sum_{0\leqslant i<j\leqslant k}(-1)^{i-1}c^k([\alpha_0],\ldots,\widehat{[\alpha_i]},\ldots,[\alpha_{j-1}],[[\![\alpha_i,\alpha_j]\!]],[\alpha_{j+1}],\ldots,[\alpha_k])\end{aligned} \tag{4.1}$$

for all $c^k \in C^k(\Omega^{n-1}(M)/\ker \#_{n-1}; C^\infty(M,\mathbb{R}))$, and $[\alpha_0],\ldots,[\alpha_k] \in \frac{\Omega^{n-1}(M)}{\ker \#_{n-1}}$.

The cohomology of this complex is called *the Nambu–Poisson cohomology* and denoted by $H^*_{NP}(M)$.

Remark 4.1. Let (M,Λ) be a Nambu–Poisson manifold of order n, $n \geqslant 3$. Consider $(C^*(\Omega^{n-1}(M); C^\infty(M,\mathbb{R})), \partial)$ the cohomology complex associated with the Leibniz algebroid $(\Lambda^{n-1}(T^*M), [\![\,,\,]\!], \#_{n-1})$. The natural projection $p : \Omega^{n-1}(M) \to \frac{\Omega^{n-1}(M)}{\ker \#_{n-1}}$ allows us to define the homomorphisms of $C^\infty(M,\mathbb{R})$-modules

$$p^k : C^k(\Omega^{n-1}(M)/\ker \#_{n-1}; C^\infty(M,\mathbb{R})) \to C^k(\Omega^{n-1}(M); C^\infty(M,\mathbb{R})) \qquad c^k \mapsto p^k(c^k)$$

$p^k(c^k) : \Omega^{n-1}(M) \times \cdots^{(k} \cdots \times \Omega^{n-1}(M) \to C^\infty(M,\mathbb{R})$ being the mapping given by

$$p^k(c^k)(\alpha_1,\ldots,\alpha_k) = c^k([\alpha_1],\ldots,[\alpha_k]).$$

A direct computation, using (2.9) and (4.1), proves that these homomorphisms induce a homomorphism between the complexes $(C^*(\Omega^{n-1}(M)/\ker \#_{n-1};\ C^\infty(M;\mathbb{R})), \tilde{\partial})$ and $(C^*(\Omega^{n-1}(M); C^\infty(M,\mathbb{R})), \partial)$. Therefore, we have the corresponding homomorphism in cohomology

$$p^* : H^*_{NP}(M) \to H^*(\Omega^{n-1}(M); C^\infty(M,\mathbb{R})).$$

Moreover, since the space of 0-cochains in both complexes is $C^\infty(M,\mathbb{R})$, then

$$p^1 : H^1_{NP}(M) \to H^1(\Omega^{n-1}(M); C^\infty(M,\mathbb{R}))$$

is a monomorphism.

Now, using the isomorphism of $C^\infty(M,\mathbb{R})$-modules

$$\overline{\#_{n-1}} : \Omega^{n-1}(M)/\ker \#_{n-1} \to \#_{n-1}(\Omega^{n-1}(M)) \qquad \overline{\#_{n-1}}([\alpha]) = \#_{n-1}(\alpha) \tag{4.2}$$

we will relate the Nambu–Poisson cohomology with the foliated cohomology of $(M,\mathcal{D})$, where $\mathcal{D}$ is the characteristic foliation of M.

The foliated cohomology of $(M,\mathcal{D})$ is defined as follows. We consider the space $\Omega^k(M,\mathcal{D})$ of the k-forms α on M such that

$$\alpha(X_1,\ldots,X_k) = 0, \qquad \text{for all} \quad X_1,\ldots,X_k \in \#_{n-1}(\Omega^{n-1}(M)).$$

From (2.8), it follows that if $\alpha \in \Omega^k(M,\mathcal{D})$ then $\mathrm{d}\alpha \in \Omega^{k+1}(M,\mathcal{D})$. Now, denote by $\Omega^k(\mathcal{D})$ the $C^\infty(M,\mathbb{R})$-module $\frac{\Omega^k(M)}{\Omega^k(M,\mathcal{D})}$. Then, the exterior differential induces a cohomology operator $\tilde{\mathrm{d}} : \Omega^k(\mathcal{D}) \to \Omega^{k+1}(\mathcal{D})$

$$\tilde{\mathrm{d}}([\alpha]) = [\mathrm{d}\alpha] \qquad \text{for} \quad [\alpha] \in \Omega^k(\mathcal{D}). \tag{4.3}$$

The resultant cohomology $H^*(\mathcal{D})$ is called the *foliated cohomology* of $(M,\mathcal{D})$ and the operator $\tilde{\mathrm{d}}$ is called the *foliated differential* of $(M,\mathcal{D})$. Note that if M is a regular Nambu–Poisson manifold, $H^*(\mathcal{D})$ is just the usual foliated cohomology of $(M,\mathcal{D})$ (see [13,19,35,36]).

On the other hand, we have:

Proposition 4.2. *Let (M,Λ) be a Nambu–Poisson manifold of order n, with $n \geqslant 3$. Then,*

$$\Omega^k(M,\mathcal{D}) = \ker \#_k$$

for all $k \in \{0,\ldots,n\}$. Thus,

$$\#_{k+1}(\mathrm{d}\alpha) = 0$$

for all $\alpha \in \ker \#_k$.

Proof. Suppose that $\alpha \in \Omega^k(M, \mathcal{D})$. We will prove that $\#_k(\alpha)(x) = 0$, for all $x \in M$.

We distinguish two cases.

(i) If $\Lambda(x) = 0$, it is clear that $\#_k(\alpha)(x) = 0$.

(ii) If $\Lambda(x) \neq 0$ then, using theorem 2.3, we deduce that there exist local coordinates $(x^1, \dots, x^n, x^{n+1}, \dots, x^m)$ in an open neighbourhood U of x such that

$$\Lambda = \frac{\partial}{\partial x^1} \wedge \cdots \wedge \frac{\partial}{\partial x^n}.$$

Now, we consider an $(n-1)$-form β_i on M satisfying

$$\#_{n-1}(\beta_i)(x) = \frac{\partial}{\partial x^i}_{|x}$$

for all $i \in \{1, \dots, n\}$. Since $\alpha \in \Omega^k(M, \mathcal{D})$, it follows that

$$\alpha(\#_{n-1}(\beta_{i_1}), \dots, \#_{n-1}(\beta_{i_k})) = 0$$

for all $1 \leqslant i_1 < \cdots < i_k \leqslant n$. Thus,

$$\alpha_x\left(\frac{\partial}{\partial x^{i_1}}_{|x}, \dots, \frac{\partial}{\partial x^{i_k}}_{|x}\right) = 0.$$

This implies that $\#_k(\alpha)(x) = 0$. Therefore, $\Omega^k(M, \mathcal{D}) \subseteq \ker \#_k$.

The proof of the inclusion $\ker \#_k \subseteq \Omega^k(M, \mathcal{D})$ is similar, using again theorem 2.3. □

In order to relate the Nambu–Poisson cohomology of a Nambu–Poisson manifold (M, Λ) of order n, $n \geqslant 3$, with the foliated cohomology of $(M, \mathcal{D})$, we introduce the monomorphisms of $C^\infty(M, \mathbb{R})$-modules

$$\tilde{i}^k : \Omega^k(\mathcal{D}) \to C^k(\Omega^{n-1}(M)/\ker \#_{n-1}; C^\infty(M, \mathbb{R})) \qquad [\alpha] \mapsto \tilde{i}^k([\alpha]) = \psi_\alpha \tag{4.4}$$

where $\psi_\alpha : \Omega^{n-1}(M)/\ker \#_{n-1} \times \cdots^{(k} \cdots \times \Omega^{n-1}(M)/\ker \#_{n-1} \to C^\infty(M, \mathbb{R})$ is the mapping given by

$$\psi_\alpha([\alpha_1], \dots, [\alpha_k]) = \alpha(\overline{\#_{n-1}}([\alpha_1]), \dots, \overline{\#_{n-1}}([\alpha_k])). \tag{4.5}$$

A direct computation, using (2.8), (4.1), (4.3)–(4.5), proves that

$$\tilde{i}^{k+1} \circ \tilde{\mathrm{d}} = \tilde{\partial} \circ \tilde{i}^k.$$

Hence, the mappings $\tilde{i}^k$ induce a monomorphism between the complexes $(\Omega^*(\mathcal{D}), \tilde{\mathrm{d}})$ and $(C^*(\Omega^{n-1}(M)/\ker \#_{n-1}; C^\infty(M, \mathbb{R})), \tilde{\partial})$.

We will denote by

$$\widetilde{\tilde{i}^k} : H^k(\mathcal{D}) \to H^k_{NP}(M)$$

the corresponding homomorphism in cohomology.

Remark 4.3. Let (M, Λ) be a regular Nambu–Poisson manifold of order n, with $n \geqslant 3$.

(i) The triple $(\frac{\Lambda^{n-1}(T^*M)}{\ker \#_{n-1}}, [\![\, ,]\!]^{\sim}, \widetilde{\#_{n-1}})$ is a Lie algebroid over M (see remark 3.2) and the Lie algebroid cohomology is just the Nambu–Poisson cohomology.

(ii) Let $\mathcal{F}$ be a foliation on a manifold N and $F = \bigcup_{n\in N} \mathcal{F}(x) \to N$ the corresponding vector subbundle of TN. Then, the mapping

$$\pi^k : \Omega^k(\mathcal{F}) = \frac{\Omega^k(N)}{\Omega^k(N,\mathcal{F})} \to C^k(\Gamma(F); C^\infty(N,\mathbb{R}))$$

defined by

$$\pi^k[\alpha](X_1,\ldots,X_k) = \alpha(X_1,\ldots,X_k)$$

for all $[\alpha] \in \Omega^k(F)$ and $X_1,\ldots,X_k \in \Gamma(F)$ is an isomorphism of $C^\infty(N,\mathbb{R})$-modules. This isomorphism induces an isomorphism between the foliated cohomology of $(N,\mathcal{F})$ and the Lie algebroid cohomology of $(F, [\,,\,], i)$, $i : F \to TN$ being the natural inclusion.

Using remarks 2.4 and 4.3, we deduce the following result:

Theorem 4.4. *Let (M,Λ) be a regular Nambu–Poisson manifold of order n, with $n \geqslant 3$. Then, the homomorphisms of $C^\infty(M,\mathbb{R})$-modules*

$$\tilde{i}^k : \Omega^k(\mathcal{D}) \to C^k\left(\frac{\Omega^{n-1}(M)}{\ker \#_{n-1}}; C^\infty(M,\mathbb{R})\right)$$

induce an isomorphism of complexes $\tilde{i}^* : (\Omega^*(\mathcal{D}), \tilde{\mathrm{d}}) \to (C^*(\frac{\Omega^{n-1}(M)}{\ker \#_{n-1}}; C^\infty(M,\mathbb{R})), \tilde{\partial})$. Thus, the Nambu–Poisson cohomology of M is isomorphic to the foliated cohomology of $(M,\mathcal{D})$, that is,

$$H^k(\mathcal{D}) \cong H^k_{NP}(M) \qquad \textit{for all } k.$$

5. A homology associated with an oriented Nambu–Poisson manifold

Let M be an m-dimensional oriented manifold and ν be a volume form on M. Denote by $\flat_\nu : \mathcal{V}^k(M) \to \Omega^{m-k}(M)$ the isomorphism of $C^\infty(M,\mathbb{R})$-modules given by

$$\flat_\nu(P) = i(P)\nu \tag{5.1}$$

for all $P \in \mathcal{V}^k(M)$.

Using this isomorphism and the exterior differential d we can define a homology operator δ_ν as follows:

$$\delta_\nu = \flat_\nu^{-1} \circ \mathrm{d} \circ \flat_\nu : \mathcal{V}^k(M) \to \mathcal{V}^{k-1}(M). \tag{5.2}$$

Note that

$$\delta_\nu(X) = \mathrm{div}_\nu X \tag{5.3}$$

for $X \in \mathfrak{X}(M)$, where $\mathrm{div}_\nu X$ is the divergence of the vector field X with respect to ν, that is, the C^∞-real valued function on M which satisfies

$$\mathcal{L}_X \nu = (\mathrm{div}_\nu X)\nu. \tag{5.4}$$

The homology associated with the complex $(\mathcal{V}^*(M), \delta_\nu)$ is denoted by $H^\nu_*(M)$ and it is dual of the de Rham cohomology of M, that is,

$$H^\nu_k(M) \cong H^{m-k}_{\mathrm{d}R}(M)$$

where $H^*_{\mathrm{d}R}(M)$ is the de Rham cohomology of M. Therefore, $H^\nu_*(M)$ does not depend of the chosen volume form.

In order to obtain an explicit expression of the operator δ_ν, we will prove the following lemma which will be useful in the following.

Lemma 5.1. *Let M be an m-dimensional oriented manifold and ν be a volume form on M. Then, for all $P \in \mathcal{V}^k(M)$ and $X \in \mathfrak{X}(M)$, we have*

$$\mathcal{L}_X \flat_\nu(P) = \flat_\nu(\mathcal{L}_X P) + (\mathrm{div}_\nu X)\flat_\nu(P). \tag{5.5}$$

Proof. If $k = 0$ or 1, relation (5.5) follows using (5.1), (5.4) and the properties of the Lie derivative operator.

Proceeding by induction on k, we deduce that (5.5) holds for a decomposable k-vector. This concludes the proof. □

Now, using this result we prove the following:

Proposition 5.2. *Let M be an m-dimensional oriented manifold and ν be a volume form on M. Then*

$$i(\alpha)\delta_\nu(P) = \mathrm{div}_\nu(i(\alpha)(P)) + (-1)^k i(\mathrm{d}\alpha)P \tag{5.6}$$

for all $P \in \mathcal{V}^k(M)$ and $\alpha \in \Omega^{k-1}(M)$.

Proof. We will proceed by induction on k.

If $k = 1$, equation (5.6) is an immediate consequence of (5.3) and (5.4).

Next, we will assume that (5.6) is true for $P \in \mathcal{V}^{k-1}(M)$ and $\alpha \in \Omega^{k-2}(M)$ and we will prove that (5.6) also holds for a decomposable k-vector P,

$$P = X_1 \wedge \cdots \wedge X_k$$

with $X_1, \ldots, X_k \in \mathfrak{X}(M)$. From (5.2),

$$\begin{aligned} \mathrm{d}(\flat_\nu(P)) &= \mathrm{d}(i(X_k)(\flat_\nu(X_1 \wedge \cdots \wedge X_{k-1}))) \\ &= \mathcal{L}_{X_k}\flat_\nu(X_1 \wedge \cdots \wedge X_{k-1}) - i(X_k)\flat_\nu(\delta_\nu(X_1 \wedge \cdots \wedge X_{k-1})). \end{aligned} \tag{5.7}$$

Now, using the induction hypothesis, we have

$$\begin{aligned} i(\beta)(\delta_\nu(X_1 \wedge \cdots \wedge X_{k-1})) = &\sum_{j=1}^{k-1}(-1)^{j+k-1}\,\mathrm{div}_\nu(\beta(X_1, \ldots, \widehat{X_j}, \ldots, X_{k-1})X_j) \\ &+(-1)^{k-1}\,\mathrm{d}\beta(X_1, \ldots, X_{k-1}) \end{aligned} \tag{5.8}$$

for all $\beta \in \Omega^{k-2}(M)$. Thus, one deduces that

$$\begin{aligned} (-1)^{k-1}\delta_\nu(X_1 \wedge \cdots \wedge X_{k-1}) = &\sum_{j=1}^{k-1}(-1)^j(\mathrm{div}_\nu(X_j))X_1 \wedge \cdots \wedge \widehat{X_j} \wedge \cdots \wedge X_{k-1} \\ &+ \sum_{1\leqslant i<j\leqslant k-1}(-1)^{i+j}[X_i, X_j] \wedge X_1 \wedge \cdots \wedge \widehat{X_i} \wedge \cdots \wedge \widehat{X_j} \wedge \cdots \wedge X_{k-1}. \end{aligned} \tag{5.9}$$

Substituting (5.9) into (5.7) and using lemma 5.1, we obtain that

$$\begin{aligned} (-1)^k d(\flat_\nu(P)) = \flat_\nu\Bigg(&\sum_{i=1}^{k}(-1)^i(\mathrm{div}_\nu X_i)X_1 \wedge \cdots \wedge \widehat{X_i} \wedge \cdots \wedge X_k \\ &+ \sum_{1\leqslant i<j\leqslant k}(-1)^{i+j}[X_i, X_j] \wedge X_1 \cdots \wedge \widehat{X_i} \wedge \cdots \wedge \widehat{X_j} \cdots \wedge X_k\Bigg). \end{aligned}$$

Consequently,

$$(-1)^k \delta_\nu(P) = \sum_{i=1}^{k} (-1)^i (\mathrm{div}_\nu(X_i)) X_1 \wedge \cdots \wedge \widehat{X_i} \wedge \cdots \wedge X_k + \sum_{1 \leqslant i<j \leqslant k} (-1)^{i+j} [X_i, X_j] \wedge X_1 \wedge \cdots \wedge \widehat{X_i} \wedge \cdots \wedge \widehat{X_j} \wedge \cdots \wedge X_k. \tag{5.10}$$

On the other hand, for all $\alpha \in \Omega^{k-1}(M)$, one has

$$(-1)^k \mathrm{div}_\nu(i(\alpha)(P)) + i(\mathrm{d}\alpha)(P) = \sum_{i=1}^{k} (-1)^i \alpha(X_1, \ldots, \widehat{X_i}, \ldots, X_k) \,\mathrm{div}_\nu X_i + \sum_{1 \leqslant i<j \leqslant k} (-1)^{i+j} \alpha([X_i, X_j], X_1, \ldots, \widehat{X_i}, \ldots, \widehat{X_j}, \ldots, X_k). \tag{5.11}$$

Therefore, from (5.10) and (5.11), we conclude that (5.6) holds for $P = X_1 \wedge \cdots \wedge X_k$ and for all $\alpha \in \Omega^{k-1}(M)$. Finally, using this result, it is easy to prove that (5.6) holds for all $P \in \mathcal{V}^k(M)$ and for all $\alpha \in \Omega^{k-1}(M)$. □

In the following, we will describe an interesting subcomplex of the complex $(\mathcal{V}^*(M), \delta_\nu)$ when M is a Nambu–Poisson manifold.

Let (M, Λ) be an m-dimensional Nambu–Poisson manifold of order n, with $n \geqslant 3$. For all $k \in \{1, \ldots, n\}$, we consider the subspace of $\mathcal{V}^k(M)$ given by

$$\mathcal{V}_t^k(M, \Lambda) = \{P \in \mathcal{V}^k(M) / i(\alpha)(P) = 0, \text{ for all } \alpha \in \Omega^1(M), \alpha \in \ker \#_1\}.$$

We will assume that $\mathcal{V}_t^0(M, \Lambda) = C^\infty(M, \mathbb{R})$.

Note that if M is a regular Nambu–Poisson manifold, $\mathcal{V}_t^k(M, \Lambda)$ is just the space of the k-vectors on M which are tangent to the characteristic foliation (see remark 2.4). Thus,

Lemma 5.3. *Let M be a regular Nambu–Poisson manifold of order n, with $n \geqslant 3$. Then*

$$\mathcal{V}_t^k(M, \Lambda) = \#_{n-k}(\Omega^{n-k}(M)) \tag{5.12}$$

for all $k \in \{0, \ldots, n\}$.

Remark 5.4. If M is an arbitrary Nambu–Poisson manifold of order n, with $n \geqslant 3$, we have that

$$\#_{n-k}(\Omega^{n-k}(M)) \subseteq \mathcal{V}_t^k(M, \Lambda) \qquad \text{for all} \quad k \in \{0, \ldots, n\}.$$

However, in general, equation (5.12) does not hold as shown by the following simple example. Suppose that M is an oriented manifold of dimension $m \geqslant 3$ and that ν is a volume form on M. Suppose also that f is a C^∞-real valued function on M such that $f^{-1}(0)$ is a finite subset of M, $f^{-1}(0) \neq \emptyset$. Denote by Λ_ν the regular Nambu–Poisson structure induced by the volume form ν. Then, the m-vector $\Lambda = f\Lambda_\nu$ defines a singular Nambu–Poisson structure of order m on M. Moreover, a direct computation proves that $\mathcal{V}_t^k(M, \Lambda) = \mathcal{V}^k(M)$ for all $k \in \{0, \ldots, m\}$. On the other hand, it is clear that if $P \in \#_{m-k}(\Omega^{m-k}(M))$ and $x \in f^{-1}(0)$ then $P(x) = 0$. Thus,

$$\#_{n-k}(\Omega^{n-k}(M)) \neq \mathcal{V}_t^k(M, \Lambda) = \mathcal{V}^k(M)$$

for all $k \in \{0, \ldots, m\}$.

Next, we will prove that if M is an oriented Nambu–Poisson manifold of order n, with $n \geqslant 3$, and ν is a volume form on M then $(\mathcal{V}_t^*(M, \Lambda) = \bigoplus_{k=1,\ldots,n} \mathcal{V}_t^k(M, \Lambda))$ is a subcomplex of the complex $(\mathcal{V}^*(M), \delta_\nu)$.

Proposition 5.5. *Let (M, Λ) be an oriented Nambu–Poisson manifold of order n, with $n \geqslant 3$, and ν be a volume form on M. Then*

$$\delta_\nu(\mathcal{V}_t^k(M, \Lambda)) \subseteq \mathcal{V}_t^{k-1}(M, \Lambda)$$

for all $k \in \{1, \ldots, n\}$.

Proof. Let α be an 1-form on M such that $\alpha \in \ker \#_1$. If $P \in \mathcal{V}_t^k(M, \Lambda)$ then, from (5.6), we have

$$\begin{aligned} i(\alpha)\delta_\nu(P)(\alpha_1, \ldots, \alpha_{k-2}) = {} & \operatorname{div}_\nu(i(\alpha \wedge \alpha_1 \wedge \cdots \wedge \alpha_{k-2})(P)) \\ & +(-1)^k i(\mathrm{d}(\alpha \wedge \alpha_1 \wedge \cdots \wedge \alpha_{k-2})(P)) \end{aligned} \tag{5.13}$$

for all $\alpha_1, \ldots, \alpha_{k-2} \in \Omega^1(M)$.

Since $\alpha \in \ker \#_1$ and $P \in \mathcal{V}_t^k(M, \Lambda)$, we obtain that

$$\begin{aligned} & i(\alpha \wedge \alpha_1 \wedge \cdots \wedge \alpha_{k-2})(P) = i(\alpha_1 \wedge \cdots \wedge \alpha_{k-2})(i(\alpha)(P)) = 0 \\ & i(d(\alpha \wedge \alpha_1 \wedge \cdots \wedge \alpha_{k-2}))(P) \\ & \qquad = i(\alpha_1 \wedge \cdots \wedge \alpha_{k-2})(i(\mathrm{d}\alpha)(P)) - i(\mathrm{d}(\alpha_1 \wedge \cdots \wedge \alpha_{k-2}))(i(\alpha)(P)) \\ & \qquad = i(\alpha_1 \wedge \cdots \wedge \alpha_{k-2})(i(\mathrm{d}\alpha)(P)). \end{aligned} \tag{5.14}$$

Next, we will see that $i(\mathrm{d}\alpha)(P) = 0$, which proves that $\delta_\nu(P) \in \mathcal{V}_t^{k-1}(M, \Lambda)$ (see (5.13) and (5.14)).

It is clear that the k-vector P induces two skew-symmetric $C^\infty(M, \mathbb{R})$-linear mappings

$$\widetilde{P} : \frac{\Omega^1(M)}{\ker \#_1} \times \cdots^{(k} \cdots \times \frac{\Omega^1(M)}{\ker \#_1} \to C^\infty(M, \mathbb{R})$$

$$\overline{P} : \#_1(\Omega^1(M)) \times \cdots^{(k} \cdots \times \#_1(\Omega^1(M)) \to C^\infty(M, \mathbb{R})$$

in such a way that

$$P(\alpha_1, \ldots, \alpha_k) = \tilde{P}([\alpha_1], \ldots, [\alpha_k]) = \overline{P}(\#_1(\alpha_1), \ldots, \#_1(\alpha_k)) \tag{5.15}$$

for all $\alpha_1, \ldots, \alpha_k \in \Omega^1(M)$. Moreover, it is easy to prove that $\overline{P}$ is a local operator, that is, if U is an open subset of M and $Q_1 \in \#_1(\Omega^1(M))$ is such that $(Q_1)_{|U} \equiv 0$ then

$$\overline{P}(Q_1, Q_2, \ldots, Q_k)_{|U} \equiv 0$$

for all $Q_2, \ldots, Q_k \in \#_1(\Omega^1(M))$.

Now, denote by R the set of the regular points of Λ

$$R = \{x \in M / \Lambda(x) \neq 0\}.$$

R and its exterior, $\operatorname{Ext}(R)$, are open subsets of M. Furthermore, it is obvious that

$$\overline{P}(\#_1(\alpha_1), \ldots, \#_1(\alpha_k))_{|\operatorname{Ext}(R)} \equiv 0$$

for all $\alpha_1, \ldots, \alpha_k \in \Omega^1(M)$. Thus, from (5.15), we deduce that

$$P(y) = 0 \qquad \text{for all} \quad y \in \operatorname{Ext}(R).$$

This implies that

$$i(\mathrm{d}\alpha)(P)_{|\operatorname{Ext}(R)} \equiv 0. \tag{5.16}$$

On the other hand, the n-vector Λ induces a regular Nambu–Poisson structure of order n on R. Therefore, from lemma 5.3, we obtain that there exists an $(n-k)$-form β on R such that

$$\#_{n-k}(\beta(y)) = P(y) \qquad \text{for all} \quad y \in R.$$

Consequently, if $y \in R$

$$i(\mathrm{d}\alpha(y))(P(y)) = i(\beta(y))(\#_2(\mathrm{d}\alpha(y)))$$

and by proposition 4.2, it follows that

$$i(\mathrm{d}\alpha)(P)_{|R} \equiv 0. \tag{5.17}$$

Finally, from (5.16), (5.17) and by continuity, we conclude that $i(\mathrm{d}\alpha)(P) = 0$. □

Let (M, Λ) be an oriented Nambu–Poisson manifold of order n, with $n \geqslant 3$, and ν be a volume form on M. Then, proposition 5.5 allows us to introduce the homology complex

$$\cdots \longrightarrow \mathcal{V}_t^{k+1}(M, \Lambda) \xrightarrow{\delta_\nu} \mathcal{V}_t^{k}(M, \Lambda) \xrightarrow{\delta_\nu} \mathcal{V}_t^{k-1}(M, \Lambda) \longrightarrow \cdots$$

This complex is called the *canonical Nambu–Poisson complex of* (M, Λ). The homology of this complex is denoted by $H_*^{canNP}(M)$ and is called *the canonical Nambu–Poisson homology of* M.

Proposition 5.6. *Let* (M, Λ) *be an oriented Nambu–Poisson manifold of order n, with* $n \geqslant 3$. *The canonical Nambu–Poisson homology does not depend on the chosen volume form.*

Proof. If ν and ν' are two volume forms on M then there exists a C^∞ real-valued function f on M such that $f \neq 0$ at every point and

$$\nu' = f\nu. \tag{5.18}$$

We can suppose, without the loss of generality, that $f > 0$.

Define the isomorphisms of $C^\infty(M, \mathbb{R})$-modules

$$\Psi^k : \mathcal{V}_t^k(M, \Lambda) \to \mathcal{V}_t^k(M, \Lambda)$$

$$P \mapsto \frac{1}{f}P$$

for all $k \in \{0, \ldots, n\}$. A direct computation, using (5.1), (5.2) and (5.18), proves that

$$\delta_{\nu'} \circ \Psi^k = \Psi^{k-1} \circ \delta_\nu. \tag{5.19}$$

Hence, the mappings Ψ^k induce an isomorphism of complexes

$$\Psi^* : (\mathcal{V}_t^*(M, \Lambda), \delta_\nu) \to (\mathcal{V}_t^*(M, \Lambda), \delta_{\nu'}).$$ □

6. Duality and the modular class of a Nambu–Poisson manifold

6.1. The modular class of a Nambu–Poisson manifold

Next, we will study when there exists a duality between the canonical Nambu–Poisson homology and the Nambu–Poisson cohomology of a Nambu–Poisson manifold (M, Λ). A fundamental tool in this study is the modular class of (M, Λ) which was introduced in [21]. We recall its definition.

Let (M, Λ) be an oriented m-dimensional Nambu–Poisson manifold of order n, with $n \geqslant 3$, and ν be a volume form on M.

Consider the mapping $\mathcal{M}^{\nu}_{\Lambda} : C^{\infty}(M, \mathbb{R}) \times \cdots^{(n-1} \cdots \times C^{\infty}(M, \mathbb{R}) \to C^{\infty}(M, \mathbb{R})$ defined by

$$\mathcal{M}^{\nu}_{\Lambda}(f_1, \ldots, f_{n-1}) = \operatorname{div}_{\nu}(X_{f_1 \ldots f_{n-1}}) \tag{6.1}$$

for all $f_1, \ldots, f_{n-1} \in C^{\infty}(M, \mathbb{R})$. Then $\mathcal{M}^{\nu}_{\Lambda}$ is a skew-symmetric $(n-1)$-linear mapping and a derivation in each argument with respect to the usual product of functions. Thus, $\mathcal{M}^{\nu}_{\Lambda}$ induces an $(n-1)$-vector on M which we also denote by $\mathcal{M}^{\nu}_{\Lambda}$.

Moreover, the mapping

$$\mathcal{M}^{\nu}_{\Lambda} : \Omega^{n-1}(M) \to C^{\infty}(M, \mathbb{R}) \qquad \alpha \mapsto i(\alpha)\mathcal{M}^{\nu}_{\Lambda} \tag{6.2}$$

defines a 1-cocycle in the Leibniz cohomology complex associated with the Leibniz algebroid $(\Lambda^{n-1}(T^*M), [\![\,,\,]\!], \#_{n-1})$ and its cohomology class $\mathcal{M}_{\Lambda} = [\mathcal{M}^{\nu}_{\Lambda}] \in H^1(\Omega^{n-1}(M); C^{\infty}(M, \mathbb{R}))$ does not depend on the chosen volume form. This cohomology class is called the *modular class* of (M, Λ).

The following result proves that the $(n-1)$-vector $\mathcal{M}^{\nu}_{\Lambda}$ defines also a 1-cocycle in the Nambu–Poisson cohomology complex.

Proposition 6.1. *Let (M, Λ) be an oriented m-dimensional Nambu–Poisson manifold of order n, with $n \geqslant 3$, and ν be a volume form on M. Then, the mapping*

$$\widetilde{\mathcal{M}^{\nu}_{\Lambda}} : \Omega^{n-1}(M)/\ker \#_{n-1} \to C^{\infty}(M, \mathbb{R}) \qquad [\alpha] \mapsto i(\alpha)\mathcal{M}^{\nu}_{\Lambda} \tag{6.3}$$

defines a 1-cocycle in the Nambu–Poisson cohomology complex of (M, Λ). Moreover, the cohomology class $\widetilde{\mathcal{M}}_{\Lambda} = [\widetilde{\mathcal{M}^{\nu}_{\Lambda}}] \in H^1_{NP}(M)$ does not depend on the chosen volume form.

Proof. Let α be an $(n-1)$-form on M. Then, using proposition 5.2, we have

$$\operatorname{div}_{\nu}(\#_{n-1}(\alpha)) = i(\alpha)\delta_{\nu}(\Lambda) + (-1)^{n-1}\#_n(\mathrm{d}\alpha). \tag{6.4}$$

Now, from (2.4), (6.1) and proposition 5.2, it follows that

$$\mathcal{M}^{\nu}_{\Lambda} = \delta_{\nu}(\Lambda). \tag{6.5}$$

Thus, using (6.4), (6.5) and proposition 4.2, we deduce that the mapping $\widetilde{\mathcal{M}^{\nu}_{\Lambda}}$ is well defined.

On the other hand, since $\mathcal{M}^{\nu}_{\Lambda}$ defines a 1-cocycle in the Leibniz cohomology complex associated with the Leibniz algebroid $(\Lambda^{n-1}(T^*M), [\![\,,\,]\!], \#_{n-1})$ then

$$i([\![\alpha, \beta]\!])\mathcal{M}^{\nu}_{\Lambda} = \#_{n-1}(\alpha)(i(\beta)\mathcal{M}^{\nu}_{\Lambda}) - \#_{n-1}(\beta)(i(\alpha)\mathcal{M}^{\nu}_{\Lambda})$$

for all $\alpha, \beta \in \Omega^{n-1}(M)$. Therefore, we conclude that (see (4.1)),

$$\tilde{\partial}\widetilde{\mathcal{M}^{\nu}_{\Lambda}}([\alpha], [\beta]) = \#_{n-1}(\alpha)(i(\beta)\mathcal{M}^{\nu}_{\Lambda}) - \#_{n-1}(\beta)(i(\alpha)\mathcal{M}^{\nu}_{\Lambda}) - i([\![\alpha, \beta]\!])\mathcal{M}^{\nu}_{\Lambda} = 0.$$

Finally, since the modular class of M does not depend on the chosen volume form, we deduce that the same is true for the cohomology class $\widetilde{\mathcal{M}}_{\Lambda} \in H^1_{NP}(M)$. □

Remark 6.2. Let (M, Λ) be an oriented Nambu–Poisson manifold of order n, with $n \geqslant 3$ and let $p^* : H^*_{NP}(M) \to H^*(\Omega^{n-1}(M); C^\infty(M, \mathbb{R}))$ be the induced homomorphism between the Nambu–Poisson cohomology of M and the Leibniz algebroid cohomology of $(\Lambda^{n-1}(T^*M), [\![\,,\,]\!], \#_{n-1})$ (see remark 4.1). Then, a direct computation, using (6.2) and (6.3), proves that

$$p^1(\widetilde{\mathcal{M}}_\Lambda) = \mathcal{M}_\Lambda.$$

Thus, since $p^1 : H^1_{NP}(M) \to H^1(\Omega^{n-1}(M); C^\infty(M, \mathbb{R}))$ is a monomorphism, it follows that the modular class of (M, Λ) is null if and only if $\widetilde{\mathcal{M}}_\Lambda = 0$.

For a regular Nambu–Poisson manifold, we have:

Theorem 6.3. *Let (M, Λ) be an oriented m-dimensional regular Nambu–Poisson manifold of order n, with $n \geqslant 3$. Then the modular class of (M, Λ) is null if and only if there exists a basic volume with respect to the characteristic foliation $\mathcal{D}$, that is, there exists $\mu \in \Omega^{m-n}(M)$ such that $\mu \neq 0$ at every point of M and*

$$i(X_{f_1\ldots f_{n-1}})\mu = 0 \qquad \mathcal{L}_{X_{f_1\ldots f_{n-1}}}\mu = 0$$

for all $f_1, \ldots, f_{n-1} \in C^\infty(M, \mathbb{R})$.

Proof. Let ν be a volume form on M and suppose that the modular class of M is null. Then, there exists $f \in C^\infty(M, \mathbb{R})$ such that

$$\mathcal{M}^\nu_\Lambda = (-1)^{n-1}\#_1(\mathrm{d}f).$$

Therefore,

$$\mathcal{M}^\nu_\Lambda(\mathrm{d}f_1, \ldots, \mathrm{d}f_{n-1}) = X_{f_1\ldots f_{n-1}}(f). \tag{6.6}$$

Taking the volume form $\nu' = e^{-f}\nu$ and using (5.4), (6.1) and (6.6), we deduce that

$$\mathcal{M}^{\nu'}_\Lambda = 0. \tag{6.7}$$

Now, we consider the $(m-n)$-form $\mu = i(\Lambda)(\nu') = \flat_{\nu'}(\Lambda)$. Then, $\mu \neq 0$ at every point of M and

$$i(X_{f_1\ldots f_{n-1}})\mu = \flat_{\nu'}(\Lambda \wedge X_{f_1\ldots f_{n-1}}) = 0.$$

Moreover, from (2.5), (6.1), (6.7) and lemma 5.1 we conclude that

$$\begin{aligned}\mathcal{L}_{X_{f_1\ldots f_{n-1}}}\mu &= \mathcal{L}_{X_{f_1\ldots f_{n-1}}}\flat_{\nu'}(\Lambda)\\ &= \flat_{\nu'}(\mathcal{L}_{X_{f_1\ldots f_{n-1}}}\Lambda) + (\mathrm{div}_{\nu'} X_{f_1\ldots f_{n-1}})\flat_{\nu'}(\Lambda) = 0.\end{aligned}$$

Conversely, suppose that there exists a basic volume μ with respect to $\mathcal{D}$. Then,

$$i(X_{f_1\ldots f_{n-1}})\mu = 0 \qquad \mathcal{L}_{X_{f_1\ldots f_{n-1}}}\mu = 0 \tag{6.8}$$

for all $f_1, \ldots, f_{n-1} \in C^\infty(M, \mathbb{R})$.

Let $D = \cup_{x\in M}\mathcal{D}(x) \to M$ be the vector subbundle of $TM \to M$ associated with $\mathcal{D}$ and $\tilde{\alpha}$ the section of the vector bundle $\Lambda^n D^* \to M$ defined as follows. If $X_1, \ldots, X_n \in \Gamma(D)$, $\tilde{\alpha}(X_1, \ldots, X_n)$ is the C^∞-real valued function on M characterized by

$$X_1 \wedge \cdots \wedge X_n = \tilde{\alpha}(X_1, \ldots, X_n)\Lambda.$$

Now, we extend $\tilde{\alpha}$ to an n-form α on M such that

$$\alpha(X_1, \ldots, X_n) = \tilde{\alpha}(X_1, \ldots, X_n)$$

for $X_1, \ldots, X_n \in \Gamma(\mathcal{D})$. It is clear that

$$i(\Lambda)(\alpha) = 1. \tag{6.9}$$

Next, we consider the volume form ν on M given by

$$\nu = \alpha \wedge \mu.$$

From (6.8) and (6.9) we have that

$$\flat_\nu(\Lambda) = \mu. \tag{6.10}$$

Thus, using (6.1), (6.8), (6.10), lemma 5.1 and the fact that $\mu \neq 0$ at every point, we conclude that

$$\mathcal{M}^\nu_\Lambda = 0. \qquad \square$$

Example 6.4. (i) Suppose that N and P are oriented manifolds and that ν (respectively, μ) is a volume form on N (respectively, P). Denote by Λ_ν the Nambu–Poisson structure on N induced by the volume form ν (see example 2.2). Λ_ν defines a regular Nambu–Poisson structure on the product manifold $M = N \times P$ and, from theorem 6.3, it follows that the modular class of (M, Λ_ν) is zero. In fact, a direct computation proves that $\mathcal{M}^{\nu\wedge\mu}_{\Lambda_\nu} = \mathcal{M}^{\nu}_{\Lambda_\nu} = 0$ and therefore (see [21]) $\mathcal{M}^{\nu\wedge\mu}_{\Lambda_\nu} = 0$. In the same way, for a function $f \in C^\infty(P, \mathbb{R})$ with zeros, $f\Lambda_\nu$ defines a singular Nambu–Poisson structure on the product manifold M and

$$\mathcal{M}^{\nu\wedge\mu}_{f\Lambda_\nu} = f\mathcal{M}^{\nu\wedge\mu}_{\Lambda_\nu} + (-1)^{n-1} i(\mathrm{d}f)(\Lambda_\nu) = 0.$$

(ii) Let $(\mathfrak{g}, [\ ,\ ])$ be the simple Lie algebra of dimension three with basis $\{\xi, \eta, \sigma\}$ satisfying

$$[\xi, \eta] = -2\eta \qquad [\xi, \sigma] = 2\sigma \qquad [\eta, \sigma] = \xi.$$

We consider a connected, simply connected, non-compact, simple Lie group G such that the Lie algebra of G is $(\mathfrak{g}, [\ ,\ ])$. From the basis $\{\xi, \eta, \sigma\}$ one can obtain a basis of left-invariant vector fields $\{\tilde{X}, \tilde{Y}, \tilde{Z}\}$ on G and if $\{\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}\}$ is the dual basis of 1-forms, we have that

$$\mathrm{d}\tilde{\alpha} = \tilde{\gamma} \wedge \tilde{\beta} \qquad \mathrm{d}\tilde{\beta} = 2\tilde{\alpha} \wedge \tilde{\beta} \qquad \mathrm{d}\tilde{\gamma} = -2\tilde{\alpha} \wedge \tilde{\gamma}.$$

Now, suppose that S is a discrete subgroup such that the space $N = S\backslash G$ of right cosets is a compact manifold (see section 4 of chapter II in [2]). Then, the vector fields $\{\tilde{X}, \tilde{Y}, \tilde{Z}\}$ (respectively, the 1-forms $\{\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}\}$) induce a global basis $\{X, Y, Z\}$ of vector fields on N (respectively, a global basis $\{\alpha, \beta, \gamma\}$ of 1-forms on N) and

$$\mathrm{d}\alpha = \gamma \wedge \beta \qquad \mathrm{d}\beta = 2\alpha \wedge \beta \qquad \mathrm{d}\gamma = -2\alpha \wedge \gamma.$$

Denote by Λ the 3-vector on the product manifold $M = N \times S^1$ given by

$$\Lambda = X \wedge Z \wedge E$$

where E is the dual vector field of the length element of S^1. It is easy to prove that Λ defines a regular Nambu–Poisson structure of order three on M.

The characteristic distribution $\mathcal{D}$ of (M, Λ) is the foliation on M given by $\beta = 0$. Thus, $\mathcal{D}$ is transversally orientable and the *Godbillon–Vey class* of $\mathcal{D}$ is the de Rham cohomology class $4[\alpha \wedge \gamma \wedge \beta]$ (for the definition of the Godbillon–Vey class of a transversally orientable foliation, see [34, p 29, 30]; see also [16]). It is clear that $[\alpha \wedge \gamma \wedge \beta] \neq 0$ and therefore we conclude that it is not possible to find a basic volume with respect to $\mathcal{D}$ (see [34, p 50]). Consequently, from theorem 6.3, we deduce that the modular class of (M, Λ) is not null.

Remark 6.5. Let M be an oriented manifold and $\mathcal{D}$ an oriented foliation on M of dimension $n \geqslant 3$. Suppose that $D = \bigcup_{x \in M} \mathcal{D}(x) \to M$ is the vector subbundle of $TM \to M$ associated with $\mathcal{D}$ and that Λ is a global section of the vector bundle $\Lambda^n D \to M$, $\Lambda \neq 0$ at every point. Then, Λ defines a regular Nambu–Poisson structure of order n on M and the characteristic foliation of (M, Λ) is just $\mathcal{D}$. Since M is an oriented manifold, the foliation $\mathcal{D}$ is transversally orientable. Thus, if the Godbillon–Vey class of $\mathcal{D}$ is not null, it follows that the modular class of (M, Λ) is not null.

6.2. Duality between the Nambu–Poisson cohomology and the canonical Nambu–Poisson homology

If M is an oriented Nambu–Poisson manifold of order n, with $n \geqslant 3$, and ν is a volume form on M, we will prove that, under certain conditions, one can define an interesting subcomplex of the homology complex $(\mathcal{V}^*(M), \delta_\nu)$. In addition, if the modular class of M vanishes, we will show that there exists a duality between the homology of this subcomplex and the foliated cohomology of $(M, \mathcal{D})$, where $\mathcal{D}$ is the characteristic foliation of M.

Theorem 6.6. *Let (M, Λ) be an oriented Nambu–Poisson manifold of order n, with $n \geqslant 3$, and ν be a volume form on M. Then:*

(a) $\#_*(\Omega^*(M)) = \bigoplus_{k=0}^{n} (\#_{n-k}(\Omega^{n-k}(M)))$ defines a subcomplex of the homology complex $(\mathcal{V}^*(M), \delta_\nu)$ if and only if $\mathcal{M}_\Lambda^\nu \in \#_1(\Omega^1(M))$.

(b) If $\#_*(\Omega^*(M))$ is a subcomplex of $(\mathcal{V}^*(M), \delta_\nu)$, then the homology of this subcomplex does not depend on the chosen volume form.

(c) If the modular class of (M, Λ) is null then $\#_*(\Omega^*(M))$ defines a subcomplex of the homology complex $(\mathcal{V}^*(M), \delta_\nu)$ and

$$\bar{H}_k^{canNP}(M) \cong H^{n-k}(\mathcal{D})$$

for all $k \in \{0, \ldots, n\}$, where $H^*(\mathcal{D})$ is the foliated cohomology of $(M, \mathcal{D})$ and $\bar{H}_*^{canNP}(M)$ denotes the homology of the complex $(\#_*(\Omega^*(M)), \delta_\nu)$.

Proof.

(a) From (5.6), (6.4) and (6.5), we have that

$$\begin{aligned} i(\alpha)\delta_\nu(\#_k(\beta)) &= \mathrm{div}_\nu(\#_{n-1}(\beta \wedge \alpha)) + (-1)^{n-k} \#_n(\beta \wedge \mathrm{d}\alpha) \\ &= i(\alpha)(i(\beta)\mathcal{M}_\Lambda^\nu + (-1)^{n-1} \#_{k+1}(\mathrm{d}\beta)) \end{aligned}$$

for all $\alpha \in \Omega^{n-k-1}(M)$ and $\beta \in \Omega^k(M)$. Thus,

$$\delta_\nu(\#_k(\beta)) = (-1)^{n-1} \#_{k+1}(\mathrm{d}\beta) + i(\beta)\mathcal{M}_\Lambda^\nu. \tag{6.11}$$

Therefore, $\delta_\nu(\#_k(\Omega^k(M)) \subseteq \#_{k+1}(\Omega^{k+1}(M))$ for all $k \in \{0, \ldots, n\}$ if and only if $\mathcal{M}_\Lambda^\nu \in \#_1(\Omega^1(M))$.

(b) Let ν' be another volume form on M. Then, there exists a C^∞-real valued function f such that $f \neq 0$ at every point and $\nu' = f\nu$. We can suppose, without the loss of generality, that $f > 0$. Thus, we can consider the isomorphisms

$$\Psi^k : \#_k(\Omega^k(M)) \to \#_k(\Omega^k(M)) \qquad P \mapsto \frac{1}{f} P.$$

Since $\delta_{\nu'} \circ \Psi^k = \Psi^{k-1} \circ \delta_\nu$, it follows that the complexes $(\#_*(\Omega^*(M)), \delta_\nu)$ and $(\#_*(\Omega^*(M)), \delta_{\nu'})$ are isomorphic.

(c) If the modular class of M is null, there exists $f \in C^\infty(M, \mathbb{R})$ such that (see (2.9))

$$\mathcal{M}^\nu_\Lambda = \#_1((-1)^{n-1}\,\mathrm{d}f). \tag{6.12}$$

Consequently, from (i), one deduces that $\#_*(\Omega^*(M))$ defines a subcomplex of $(\mathcal{V}^*(M), \delta_\nu)$.

On the other hand, using proposition 4.2, we can define the isomorphisms of $C^\infty(M, \mathbb{R})$-modules

$$h_k : \Omega^{n-k}(\mathcal{D}) = \Omega^{n-k}(M)/\ker \#_{n-k} \to \#_{n-k}(\Omega^{n-k}(M)) \qquad h_k([\alpha]) = e^{-f}\#_{n-k}(\alpha).$$

From (6.2), (6.11) and (6.12) it follows that $h_k \circ \tilde{\mathrm{d}} = (-1)^{n-1}\delta_\nu \circ h_{k+1}$, where $\tilde{\mathrm{d}}$ is the foliated differential of $(M, \mathcal{D})$. So, the above isomorphisms induce an isomorphism between the cohomology group $H^{n-k}(\mathcal{D})$ and the homology group $\bar{H}^{canNP}_k(M)$. □

Using remark 4.3 and theorems 6.3 and 6.6, we deduce that

Corollary 6.7. *Let (M, Λ) be an oriented regular Nambu–Poisson manifold of order n, with $n \geqslant 3$. If there exists a basic volume with respect to the characteristic foliation $\mathcal{D}$ of (M, Λ) then*

$$H^k_{NP}(M) \cong H^k(\mathcal{D}) \cong H^{canNP}_{n-k}(M)$$

for all $k \in \{0, \ldots, n\}$.

7. A singular Nambu–Poisson structure

Consider on $\mathbb{R}^3$ the 3-vector defined by

$$\Lambda = (x_1^2 + x_2^2 + x_3^2)\frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_3} \tag{7.1}$$

where (x_1, x_2, x_3) denote the usual coordinates on $\mathbb{R}^3$. The 3-vector Λ defines a singular Nambu–Poisson structure of order three on $\mathbb{R}^3$. Let ν be the volume form given by

$$\nu = \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3.$$

A direct computation proves that

$$X_{x_1x_2} = (x_1^2 + x_2^2 + x_3^2)\frac{\partial}{\partial x_3}$$
$$X_{x_1x_3} = -(x_1^2 + x_2^2 + x_3^2)\frac{\partial}{\partial x_2}$$
$$X_{x_2x_3} = (x_1^2 + x_2^2 + x_3^2)\frac{\partial}{\partial x_1}$$

and therefore (see (6.1))

$$\mathcal{M}^\nu_\Lambda = 2x_3\frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2} - 2x_2\frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_3} + 2x_1\frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_3}.$$

Now, if the modular class of $(\mathbb{R}^3, \Lambda)$ were null then there exists $f \in C^\infty(\mathbb{R}^3, \mathbb{R})$ such that

$$i(\alpha)\mathcal{M}^\nu_\Lambda = \#_2\alpha(f)$$

for all $\alpha \in \Omega^2(\mathbb{R}^3)$. Taking the 2-forms $\mathrm{d}x_1 \wedge \mathrm{d}x_2$, $\mathrm{d}x_1 \wedge \mathrm{d}x_3$, $\mathrm{d}x_2 \wedge \mathrm{d}x_3$ we would deduce that

$$2x_j = (x_1^2 + x_2^2 + x_3^2)\frac{\partial f}{\partial x_j} \qquad \text{for all} \quad j = 1, 2, 3. \tag{7.2}$$

Then,

$$f_{|\mathbb{R}^3-\{(0,0,0)\}} = \ln(x_1^2 + x_2^2 + x_3^2) + c \qquad \text{with} \quad c \in \mathbb{R}.$$

However, this is not possible because of $f \in C^\infty(\mathbb{R}^3, \mathbb{R})$. Thus, the modular class of $(\mathbb{R}^3, \Lambda)$ is not null.

Next, we will prove that there is no duality between the Nambu–Poisson cohomology and the canonical Nambu–Poisson homology of $(\mathbb{R}^3, \Lambda)$. In fact, we will show that

$$H^1_{NP}(\mathbb{R}^3) \ncong H_2^{canNP}(\mathbb{R}^3).$$

First, we compute $H^1_{NP}(\mathbb{R}^3)$. In order to do this, we will proceed as follows.

Since $\ker \#_2 = \{0\}$, then

$$\Omega^2(\mathbb{R}^3) \cong \#_2(\Omega^2(\mathbb{R}^3)) = \{(x_1^2 + x_2^2 + x_3^2)X / X \in \mathfrak{X}(\mathbb{R}^3)\}.$$

This fact implies that one can identify the co-chains $c^1 : \#_2(\Omega^2(\mathbb{R}^3)) \to C^\infty(\mathbb{R}^3, \mathbb{R})$ of the Nambu–Poisson cohomology complex with the 1-forms on $\mathbb{R}^3$ using the isomorphism:

$$\Phi : C^1(\Omega^2(\mathbb{R}^3); C^\infty(\mathbb{R}^3, \mathbb{R})) \to \Omega^1(\mathbb{R}^3) \qquad (c^1 : \Omega^2(\mathbb{R}^3) \to C^\infty(\mathbb{R}^3, \mathbb{R})) \mapsto \alpha$$

such that $\alpha(X) = c^1(\beta)$, where $\#_2(\beta) = (x_1^2 + x_2^2 + x_3^2)X$.

Under this identification the first Nambu–Poisson cohomology group $H^1_{NP}(\mathbb{R}^3)$ is the quotient space

$$\frac{\{\alpha \in \Omega^1(\mathbb{R}^3)/(x_1^2 + x_2^2 + x_3^2)\,\mathrm{d}\alpha - \mathrm{d}(x_1^2 + x_2^2 + x_3^2) \wedge \alpha = 0\}}{\{(x_1^2 + x_2^2 + x_3^2)\,\mathrm{d}g / g \in C^\infty(\mathbb{R}^3, \mathbb{R})\}}. \tag{7.3}$$

Now, we consider the set

$$\mathcal{G} = \left\{ g \in C^\infty(\mathbb{R}^3 - \{(0,0,0)\}, \mathbb{R}) / (x_1^2 + x_2^2 + x_3^2)\frac{\partial g}{\partial x_i} \in C^\infty(\mathbb{R}^3, \mathbb{R}),\ \text{for all } i \in \{1,2,3\} \right\}$$

and the linear map

$$\mathcal{T} : \mathcal{G} \to H^1_{NP}(\mathbb{R}^3)$$

defined by $\mathcal{T}(g) = [(x_1^2 + x_2^2 + x_3^3)\,\mathrm{d}g]$. It is clear that the kernel of this mapping is the space $C^\infty(\mathbb{R}^3, \mathbb{R})$. Moreover, $\mathcal{T}$ is an epimorphism. In fact, if $[\alpha] \in H^1_{NP}(\mathbb{R}^3)$, from (7.3), we deduce that in $\mathbb{R}^3 - \{(0,0,0)\}$

$$\mathrm{d}\left(\frac{\alpha}{x_1^2 + x_2^2 + x_3^2}\right) = 0.$$

But this implies that there exists $g \in C^\infty(\mathbb{R}^3 - \{(0,0,0)\}, \mathbb{R})$ such that

$$\frac{\alpha}{x_1^2 + x_2^2 + x_3^2} = \mathrm{d}g$$

and therefore

$$\mathcal{T}(g) = [\alpha].$$

Thus,

$$\frac{\mathcal{G}}{C^\infty(\mathbb{R}^3, \mathbb{R})} \cong H^1_{NP}(\mathbb{R}^3). \tag{7.4}$$

Next, we will prove that the quotient space $\frac{\mathcal{G}}{C^\infty(\mathbb{R}^3,\mathbb{R})}$ is isomorphic to $\mathbb{R}$. To do that, we will use the following lemmas (a proof of the first lemma can be found in [30]).

Lemma 7.1 (See [30]). *Let P, Q be two polynomials of degree n, $(n \geqslant 1)$ in the indeterminates x_1 and x_2 such that satisfy*

$$(x_1^2 + x_2^2)\left(\frac{\partial P}{\partial x_2} - \frac{\partial Q}{\partial x_1}\right) = 2(Px_2 - Qx_1).$$

Then there exist two polynomials $\tilde{P}$, $\tilde{Q}$ of degree $n-2$ such that P and Q are written in the following form:

$$P = ax_1 + bx_2 + (x_1^2 + x_2^2)\tilde{P} \qquad Q = bx_1 + ax_2 + (x_1^2 + x_2^2)\tilde{Q}$$

where a, b are real constants and $\frac{\partial \tilde{P}}{\partial x_2} = \frac{\partial \tilde{Q}}{\partial x_1}$.

Lemma 7.2. *Let A, B and C be three polynomials of degree n, $(n \geqslant 1)$ in the indeterminates x_1, x_2, x_3, such that satisfy*

$$\begin{aligned}
(x_1^2 + x_2^2 + x_3^2)\left(\frac{\partial A}{\partial x_2} - \frac{\partial B}{\partial x_1}\right) &= 2(Ax_2 - Bx_1)\\
(x_1^2 + x_2^2 + x_3^2)\left(\frac{\partial A}{\partial x_3} - \frac{\partial C}{\partial x_1}\right) &= 2(Ax_3 - Cx_1)\\
(x_1^2 + x_2^2 + x_3^2)\left(\frac{\partial B}{\partial x_3} - \frac{\partial C}{\partial x_2}\right) &= 2(Ax_3 - Cx_2).
\end{aligned} \tag{7.5}$$

Then there exist three polynomials $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ of degree $n-2$ such that A, B and C are written in the following form:

$$\begin{aligned}
A &= ax_1 + (x_1^2 + x_2^2 + x_3^2)\tilde{A}\\
B &= ax_2 + (x_1^2 + x_2^2 + x_3^2)\tilde{B}\\
C &= ax_3 + (x_1^2 + x_2^2 + x_3^2)\tilde{C}
\end{aligned}$$

where a is a real constant and $\frac{\partial \tilde{A}}{\partial x_2} = \frac{\partial \tilde{B}}{\partial x_1}$, $\frac{\partial \tilde{A}}{\partial x_3} = \frac{\partial \tilde{C}}{\partial x_1}$ and $\frac{\partial \tilde{B}}{\partial x_3} = \frac{\partial \tilde{C}}{\partial x_2}$.

Proof. It is sufficient to prove the result for the case when A, B and C are homogeneous polynomials. If $n = 1$ is clear that $A = ax_1$, $B = ax_2$ and $C = ax_3$. If $n \geqslant 2$ we proceed as follows.

The polynomials A and B can be written as

$$A(x_1, x_2, x_3) = \sum_{k=0}^{n} x_3^k A_k(x_1, x_2) \qquad B(x_1, x_2, x_3) = \sum_{k=0}^{n} x_3^k B_k(x_1, x_2)$$

where $A_i(x_1, x_2)$ and $B_i(x_1, x_2)$ $(i = 0, \ldots, n)$ are homogeneous polynomials in the indeterminates x_1, x_2.

From the first equality of (7.5) we deduce that

$$(x_1^2 + x_2^2)\left(\frac{\partial A_i}{\partial x_2} - \frac{\partial B_i}{\partial x_1}\right) = 2(A_i x_2 - B_i x_1) \qquad i \in \{0, 1\} \tag{7.6}$$

and for all $r \in \{2, \ldots n\}$,

$$(x_1^2 + x_2^2)\left(\frac{\partial A_r}{\partial x_2} - \frac{\partial B_r}{\partial x_1}\right) + \left(\frac{\partial A_{r-2}}{\partial x_2} - \frac{\partial B_{r-2}}{\partial x_1}\right) = 2(A_r x_2 - B_r x_1). \tag{7.7}$$

Using (7.6) and lemma 7.1 we obtain that there exist $\tilde{A}_0$, $\tilde{A}_1$, $\tilde{B}_0$ and $\tilde{B}_1$ polynomials in the indeterminates x_1, x_2 such that

$$A_i = (x_1^2 + x_2^2)\tilde{A}_i \qquad B_i = (x_1^2 + x_2^2)\tilde{B}_i \qquad \frac{\partial \tilde{A}_i}{\partial x_2} = \frac{\partial \tilde{B}_i}{\partial x_1}$$

for $i = 0, 1$.

Now, from these facts and (7.7), we have that

$$(x_1^2 + x_2^2)\left(\frac{\partial(A_2 - \tilde{A}_0)}{\partial x_2} - \frac{\partial(B_2 - \tilde{B}_0)}{\partial x_1}\right) = 2x_2(A_2 - \tilde{A}_0) - 2x_1(B_2 - \tilde{B}_0).$$

Applying again lemma 7.1 we deduce that there exist $\tilde{A}_2$ and $\tilde{B}_2$ polynomials in the indeterminates x_1 and x_2 such that

$$A_2 = \tilde{A}_0 + (x_1^2 + x_2^2)\tilde{A}_2 \qquad B_2 = \tilde{B}_0 + (x_1^2 + x_2^2)\tilde{B}_2$$

with $\frac{\partial \tilde{A}_2}{\partial x_2} = \frac{\partial \tilde{B}_2}{\partial x_1}$.

Proceeding in a similar way we obtain a sequence of polynomials $\tilde{A}_0, \ldots, \tilde{A}_n, \tilde{B}_0, \ldots, \tilde{B}_n$ in the indeterminates x_1 and x_2 such that

$$A_i = (x_1^2 + x_2^2)\tilde{A}_i \qquad B_i = (x_1^2 + x_2^2)\tilde{B}_i$$
$$A_r = \tilde{A}_{r-2} + (x_1^2 + x_2^2)\tilde{A}_r \qquad B_r = \tilde{B}_{r-2} + (x_1^2 + x_2^2)\tilde{B}_r$$

for $i \in \{0, 1\}$ and for $r \in \{2, \ldots, n\}$. Thus, the polynomials A and B can be written as

$$A = (x_1^2 + x_2^2 + x_3^2)\sum_{k=0}^{n} x_3^k \tilde{A}_k \qquad B = (x_1^2 + x_2^2 + x_3^2)\sum_{k=0}^{n} x_3^k \tilde{B}_k.$$

Using the same process we also deduce that the polynomial C can be written as

$$C = (x_1^2 + x_2^2 + x_3^2)\sum_{k=0}^{n} x_1^k \tilde{C}_k$$

where $\tilde{C}_k$ are polynomials in the indeterminates x_2 and x_3. □

This last lemma allows us to obtain the announced result.

Proposition 7.3. *The quotient space* $\frac{\mathcal{G}}{C^\infty(\mathbb{R}^3,\mathbb{R})}$ *is isomorphic to* $\mathbb{R}$.

Proof. Taking $g \in \mathcal{G}$ we have that the C^∞ real-valued functions on $\mathbb{R}^3$

$$g_1 = (x_1^2 + x_2^2 + x_3^2)\frac{\partial g}{\partial x_1} \qquad g_2 = (x_1^2 + x_2^2 + x_3^2)\frac{\partial g}{\partial x_2} \qquad g_3 = (x_1^2 + x_2^2 + x_3^2)\frac{\partial g}{\partial x_3}$$

satisfy

$$\begin{aligned}
(x_1^2 + x_2^2 + x_3^2)\left(\frac{\partial g_1}{\partial x_2} - \frac{\partial g_2}{\partial x_1}\right) &= 2(x_2 g_1 - x_1 g_2) \\
(x_1^2 + x_2^2 + x_3^2)\left(\frac{\partial g_1}{\partial x_3} - \frac{\partial g_3}{\partial x_1}\right) &= 2(x_3 g_1 - x_1 g_3) \qquad (7.8)\\
(x_1^2 + x_2^2 + x_3^2)\left(\frac{\partial g_2}{\partial x_3} - \frac{\partial g_3}{\partial x_2}\right) &= 2(x_3 g_2 - x_2 g_3).
\end{aligned}$$

Then, for arbitrary $n \geqslant 2$, let consider the Taylor expansions of order $n+1$ at the origin of the functions g_1, g_2, g_3. We write these Taylor expansions as $g_1 = A_n + R_{1,n}$, $g_2 = B_n + R_{2,n}$ and

$g_3 = C_n + R_{3,n}$ where A_n, B_n, C_n are polynomials of degree n which satisfy the conditions of lemma 7.2 and $R_{i,n}$ are the remainder terms. Denote by $[k(x_1, x_2, x_3)]_{(0,0,0)}$ the formal Taylor expansion at the origin of $k \in C^\infty(\mathbb{R}^3, \mathbb{R})$. Then there exists $a \in \mathbb{R}$ such that

$$\begin{aligned}
[g_1(x_1, x_2, x_3) - ax_1]_{(0,0,0)} &= (x_1^2 + x_2^2 + x_3^2)A(x_1, x_2, x_3) \\
[g_2(x_1, x_2, x_3) - ax_2]_{(0,0,0)} &= (x_1^2 + x_2^2 + x_3^2)B(x_1, x_2, x_3) \\
[g_3(x_1, x_2, x_3) - ax_3]_{(0,0,0)} &= (x_1^2 + x_2^2 + x_3^2)C(x_1, x_2, x_3)
\end{aligned}$$

where $A(x_1, x_2, x_3)$, $B(x_1, x_2, x_3)$ and $C(x_1, x_2, x_3)$ are suitable formal power series. Using Borel's theorem we have that there exist $\alpha, \beta, \gamma \in C^\infty(\mathbb{R}^3, \mathbb{R})$ such that

$$\begin{aligned}
[\alpha(x_1, x_2, x_3)]_{(0,0,0)} &= A(x_1, x_2, x_3) \\
[\beta(x_1, x_2, x_3)]_{(0,0,0)} &= B(x_1, x_2, x_3) \\
[\gamma(x_1, x_2, x_3)]_{(0,0,0)} &= C(x_1, x_2, x_3).
\end{aligned}$$

Note that the formal Taylor expansions at the origin of the functions

$$\begin{aligned}
\alpha_1 &= g_1 - ax_1 - (x_1^2 + x_2^2 + x_3^2)\alpha \\
\beta_1 &= g_2 - ax_2 - (x_1^2 + x_2^2 + x_3^2)\beta \\
\gamma_1 &= g_3 - ax_3 - (x_1^2 + x_2^2 + x_3^2)\gamma
\end{aligned}$$

vanish. Therefore, $\frac{\alpha_1}{(x_1^2+x_2^2+x_3^2)}$, $\frac{\beta_1}{(x_1^2+x_2^2+x_3^2)}$ and $\frac{\gamma_1}{(x_1^2+x_2^2+x_3^2)}$ are C^∞ real-valued functions on $\mathbb{R}^3$.

Let us consider the C^∞ real-valued functions on $\mathbb{R}^3$

$$h_1 = \alpha + \frac{\alpha_1}{(x_1^2 + x_2^2 + x_3^2)} \qquad h_2 = \beta + \frac{\beta_1}{(x_1^2 + x_2^2 + x_3^2)} \qquad h_3 = \gamma + \frac{\gamma_1}{(x_1^2 + x_2^2 + x_3^2)}.$$

Then, using (7.8) and the fact that

$$g_i = ax_i + (x_1^2 + x_2^2 + x_3^2)h_i \qquad i = 1, 2, 3$$

we obtain that

$$\frac{\partial h_1}{\partial x_2} - \frac{\partial h_2}{\partial x_1} = \frac{\partial h_1}{\partial x_3} - \frac{\partial h_3}{\partial x_1} = \frac{\partial h_2}{\partial x_3} - \frac{\partial h_3}{\partial x_2} = 0. \tag{7.9}$$

Therefore,

$$\mathrm{d}g = \left(\sum_{i=1}^{3} \frac{g_i}{x_1^2 + x_2^2 + x_3^2}\,\mathrm{d}x_i\right)_{|\mathbb{R}^3 - \{(0,0,0)\}} = \left(\mathrm{d}\left(\frac{a}{2}\ln(x_1^2 + x_2^2 + x_3^2)\right) + \sum_{i=1}^{3} h_i\,\mathrm{d}x_i\right)_{|\mathbb{R}^3 - \{(0,0,0)\}}. \tag{7.10}$$

On the other hand, using (7.9) we deduce that $h_1\,\mathrm{d}x_1 + h_2\,\mathrm{d}x_2 + h_3\,\mathrm{d}x_3$ is a closed 1-form on $\mathbb{R}^3$ and, since $H^1_{\mathrm{d}R}(\mathbb{R}^3) = \{0\}$, we conclude that there exists $\psi \in C^\infty(\mathbb{R}^3, \mathbb{R})$ such that $h_1\,\mathrm{d}x_1 + h_2\,\mathrm{d}x_2 + h_3\,\mathrm{d}x_3 = \mathrm{d}\psi$. Substituting in (7.10) we have that

$$g - \tfrac{1}{2}a\ln(x_1^2 + x_2^2 + x_3^2) = \psi_{|\mathbb{R}^3 - \{(0,0,0)\}} + c \qquad \text{with} \quad c \in \mathbb{R}.$$

Consequently,

$$[g] = \left[\frac{a}{2}\ln(x_1^2 + x_2^2 + x_3^2)\right] \qquad \text{with} \quad a \in \mathbb{R}.$$

This completes the proof. □

From (7.4) and proposition 7.3, we deduce that

Proposition 7.4. *Let Λ be the Nambu–Poisson structure on $\mathbb{R}^3$ given by (7.1). Then,*

$$H^1_{NP}(\mathbb{R}^3) \cong \mathbb{R}.$$

Remark 7.5. In [29] the author has generalized the above result for germs at 0 of n-vectors $\Lambda = f\frac{\partial}{\partial x_1} \wedge \cdots \wedge \frac{\partial}{\partial x_n}$ on $\mathbb{R}^n$ or $\mathbb{C}^n$, where f is a quasihomogeneous polynomial of finite codimension. In fact, in this paper all Nambu–Poisson cohomology groups for this type of structures are computed.

On the other hand, since $\ker \#_1 = \{0\}$, it follows that

$$\mathcal{V}^k_t(\mathbb{R}^3, \Lambda) = \mathcal{V}^k(\mathbb{R}^3)$$

for all k. Thus, the canonical Nambu–Poisson homology of $(\mathbb{R}^3, \Lambda)$ is dual of the de Rham cohomology. In particular, $H_2^{canNP}(\mathbb{R}^3) \cong H^1_{\mathrm{d}R}(\mathbb{R}^3) = \{0\}$.

This implies that $H^1_{NP}(\mathbb{R}^3) \ncong H_2^{canNP}(\mathbb{R}^3)$ and therefore the duality between the Nambu–Poisson cohomology and the canonical Nambu–Poisson homology does not hold.

Remark 7.6.

(i) If $\#_r : \Omega^r(\mathbb{R}^3) \to \mathcal{V}^{3-r}(\mathbb{R}^3)$, $r = 1, 2, 3$, is the induced homomorphism by the Nambu–Poisson structure Λ on $\mathbb{R}^3$, then, it is clear that $\#_r$ is a monomorphism. Therefore, if $\mathcal{D}$ is the characteristic foliation of $(\mathbb{R}^3, \Lambda)$, we have that the foliated cohomology of $(\mathbb{R}^3, \mathcal{D})$ is isomorphic to the de Rham cohomology. In particular, $H^1_{NP}(\mathbb{R}^3) \ncong H^1(\mathcal{D}) = \{0\}$. Consequently, the Nambu–Poisson cohomology and the foliated cohomology are not isomorphic.

(ii) A direct computation shows that $\mathcal{M}^\nu_\Lambda \notin \#_1(\Omega^1(\mathbb{R}^3))$. Thus, $\#_*(\Omega^*(\mathbb{R}^3)) = \bigoplus_{k=0,\ldots,3} \#_k(\Omega^k(\mathbb{R}^3))$ is not a subcomplex of the homology complex $(\mathcal{V}^*(\mathbb{R}^3), \delta_\nu)$ (see theorem 6.6).

Acknowledgments

This work has been partially supported through grants DGICYT (Spain) (Projects PB97-1257, PB97-1487 and BFM2000-0808) and Project UPV 127.310-EA147/98. In addition, we like to thank several institutions for their hospitality while work on this project was being done: the Department of Fundamental Mathematics from University of La Laguna (RI) and Department of Mathematics from University of the Basque Country (JCM, EP). We also would like to acknowledge to Nobutada Nakanishi and Marta Macho-Stadler for helpful discussions.

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